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THE SCATTERING BY A HEADLAND OF THE DOMINANT CONTINENTAL SHELF WAVE

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CONTENTS

The scattering that occurs when a continental shelf wave propagating over an exponential–constant depth profile is incident on a coastal promontory is here investigated by transforming the shelf region differential equation for the mass transport stream function to the wave equation and using high wavenumber asymptotics and the method of successive reflexions. It is found that energy is propagated in the shelf region according to geometrical optics, being reflected at the shoreline, promontory and shelf–ocean boundary. A solution is sought in terms of a distribution of singularities on the boundary of the promontory, and the application of all boundary conditions is facilitated by using plane wave representations. Numerical values obtained experimentally indicate a shelf width of some three times the wavelength which is similar to the maximum projection of the promontory from the shoreline. Thus it is plausible to seek an approximate solution by alternately considering the scattering by the obstacle with the shelf width regarded as infinite and the reflexions at the shelf–ocean boundary in the absence of the promontory. For the semicircular headland, an exact solution is available for the initial scattering but the successive approximation procedure converges less quickly than for the offshore barrier, in which case no exact solutions are available. The reason is that the tendency for energy to become trapped on the shortest offshore line across the shelf is more sharply focused in the latter case. Numerical values are given for the amplitudes of the far field modes excited by the semicircular headland. For the offshore barrier, the density of the singularity distribution corresponds to the velocity discontinuity at the barrier, while at the line of the barrier the alongshore and offshore components of the shelf region velocity field due to scattering are such that, to leading order, corresponding terms differ only by constant phase.

1. Introduction

Recently, increasing attention has been devoted to the study of coastal trapped waves whic in a homogeneous fluid, occur as low frequency continental shelf waves, high frequency ed waves and intermediate frequency Kelvin waves. Reviews of this topic are given by LeBlor & Mysak (1977) and Mysak (1980). When the rigid-lid approximation is used, the extern Kelvin wave and the surface (gravity) edge waves are eliminated (LeBlond & Mysak 197 ch. 3). In this circumstance and with the Coriolis parameter f assumed constant in mi latitude, Buchwald & Adams (1968) showed that the horizontally non-divergent linearize shallow-water equations imply the existence on an exponential shelf profile of finite width free shelf waves which propagate with the shallow water to the right or left of the alongsho phase velocity in the Northern or Southern Hemisphere respectively. For each value of th frequency $\omega < f$, a finite number of offshore modes is possible, in each of which the alongsho group velocity, on the shelf, has the same or opposite sign as the phase velocity according the alongshore wavelength is longer or shorter than a critical distance depending on σ ($= \omega/f$ and the rate of change of depth. The zero group velocity to which Buchwald (1977) refe occurs only in the ocean region. On the shelf, each shelf wave mode consists of two plane wav components whose group velocity vectors have cancelling offshore components, leaving i general a net transport of energy in the alongshore direction. It is possible to have energ trapped in the shelf region but at sets of values of the parameters differing slightly from thos corresponding to zero group velocity in the ocean. It is shown that by transforming the differen tial equation for the stream function in the shelf region to its canonical form, which is the wav equation, the group velocity vector associated with each constituent term can be identified a having direction exactly opposite to the corresponding phase velocity vector, and hence th radiation condition for scattering can be easily applied.

The goal of the present paper is an investigation of the scattering of a long shelf wave by thin barrier at right angles to the shoreline and considerably shorter than the shelf width. I is motivated by the need to understand, for the benefit of navigation and fishing, the curren motions near the Brooks Peninsula which projects from the west coast of Vancouver Island British Columbia. On inserting the numerical data, it is found that the barrier length a i slightly more than one wavelength (with respect to the transformed differential equation while the shelf width l is 2.5 times larger.

Thus, although the scatterer cannot be regarded as small, a successful analysis can be achieved by using high wavenumber asymptotics of the wave equation and the method o successive reflexions. If the finite shelf width is ignored, the scattering of the incident shel wave can be considered by standard methods. Then, on using a plane wave expansion, the application to this scattered field of the conditions at the shelf–ocean boundary and shoreline, but not the scatterer, yields waves repeatedly reflected in the shelf region and edge waves refracted in the ocean. This process is continued until sufficient accuracy is achieved.

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is only possible if h'/h is constant, corresponding to exponential dependence of h on y. Considering a west coast in the Northern Hemisphere ($f > 0$), let x be measured northward and y westward with $y = 0$ at the shoreline. Suppose that the depth profile is given by

$$\left.\begin{aligned} h &= H\mathrm{e}^{-2b(l-y)} \quad (0 \leqslant y \leqslant l), \\ h &= H \quad (y \geqslant l). \end{aligned}\right\} \tag{2.5}$$

Then (2.4) implies that

$$\nabla^2\psi = 2b\left(\frac{\partial\psi}{\partial y} + \frac{i}{\sigma}\frac{\partial\psi}{\partial x}\right) \quad (0 < y < l), \tag{2.6}$$

$$\nabla^2\psi = 0 \quad (y > l), \tag{2.7}$$

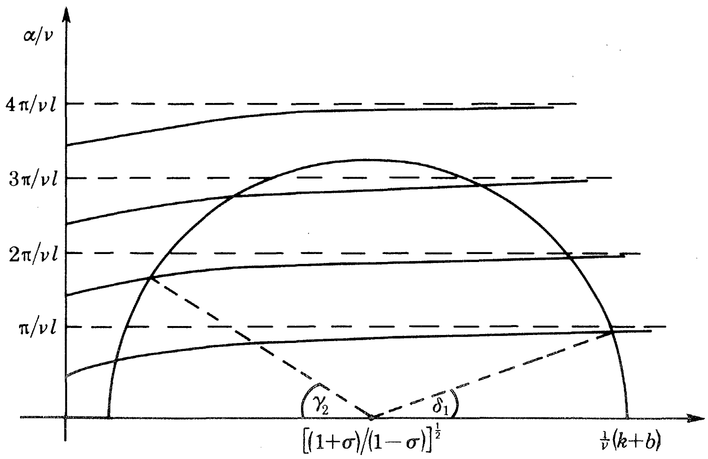

FIGURE 1. Typical curves in the k, α-plane showing how eigenvalues are determined by equations (2.12) and (2.13).

where $\sigma = \omega/f > 0$. The boundary conditions are

$$\psi = 0 \quad \text{at} \quad y = 0, \tag{2.8}$$

$$\psi \quad \text{and} \quad \partial\psi/\partial y \quad \text{continuous at} \quad y = l, \tag{2.9}$$

$$\psi \quad \text{bounded as} \quad (x^2+y^2)^{\frac{1}{2}} \to \infty. \tag{2.10}$$

Buchwald & Adams (1968) showed that shelf wave solutions, of (2.6) to (2.10) inclusive, exist of the form

$$\left.\begin{aligned} \psi &= (\sin\alpha y/\sin\alpha l)\,\mathrm{e}^{ikx+b(y-l)} \quad (0 \leqslant y \leqslant l), \\ \psi &= \mathrm{e}^{k[ix-(y-l)]} \quad (y \geqslant l) \end{aligned}\right\} \tag{2.11}$$

provided that α and k satisfy the following two equations:

$$\alpha^2 + b^2 + k^2 - 2bk/\sigma = 0, \tag{2.12}$$

$$\alpha l \cot \alpha l + (k+b)\,l = 0. \tag{2.13}$$

For real α in (2.12), it is necessary to have $k > 0$ and $\sigma < 1$. Then for each k in the range

$$0 < (b/\sigma)\left[1-(1-\sigma^2)^{\frac{1}{2}}\right] < k < (b/\sigma)\left[1+(1-\sigma^2)^{\frac{1}{2}}\right],$$

equation (2.12) yields one value of α (the negative root can be discarded) while (2.13) gives infinitely many values of α. When one of these values coincides with that obtained from (2.12),

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The solution $(\nu l)_N$ in the interval $[(N-\frac{1}{2})\pi, N\pi]$ $(N \geqslant 1)$ is given by

$$(\nu l)_N = N\pi - \tfrac{1}{2}\arccos\sigma. \tag{2.18}$$

With this regarded as a first approximation to $(\widehat{\nu l})_N$, it then follows that

$$(\widehat{\nu l})_N \sim (\nu l)_N - (1-\sigma)^2/8(\nu l)_N. \tag{2.19}$$

Thus, if $(\nu l)_N < \nu l < (\widehat{\nu l})_{N+1}$, there are N subcritical modes with

$$k = b/\sigma - \nu\cos\gamma_m, \quad \alpha = \nu\sin\gamma_m \quad (1 \leqslant m \leqslant N)$$

and N supercritical modes with

$$k = b/\sigma + \nu\cos\delta_m, \quad \alpha = \nu\sin\delta_m \quad (1 \leqslant m \leqslant N).$$

The distinct sets $\{\gamma_m\}$ and $\{\delta_m\}$ of acute angles are each labelled in ascending order, i.e.

$$(m-\tfrac{1}{2})(\pi/\nu l) < \sin\gamma_m < \sin\delta_m < (m\pi/\nu l) \quad (1 \leqslant m \leqslant N)$$

as indicated in figure 1. As $\nu l \to (\nu l)_N$ from above, $\delta_N \to \frac{1}{2}\pi$.

If $(\widehat{\nu l})_{N+1} < \nu l < (\nu l)_{N+1}$, a narrow band of values, then in addition to the $2N$ modes described above, there are two subcritical modes which, as $\nu l \to (\widehat{\nu l})_{N+1}$ from above, coalesce to form a single subcritical mode and the $(N+1)$th resonant mode.

As explained earlier, equation (2.6) can be reduced to one of standard form by writing

$$\psi(x,y) = \chi(x,y)\,\mathrm{e}^{b(y+\mathrm{i}x/\sigma)} \quad (0 \leqslant y \leqslant l), \tag{2.20}$$

whence

$$(\nabla^2+\nu^2)\chi = 0 \quad (0 < y < l), \tag{2.21}$$

where ν is given by (2.15). Note that the factor e^{by} is more than balanced by the depth change factor in (2.5), so that the velocity amplitudes have decay factor e^{-by}.

Now, since all eigenvalues of k are positive, the x-component of the phase velocity of the solution for $\psi\,\mathrm{e}^{-\mathrm{i}\omega t}$ defined by (2.11) is positive, i.e. northward travelling. However, consideration of group velocity shows that energy can still be propagated southward. Suppose that a solution of (2.21) has a wavenumber vector that makes an angle β with the positive x-axis. Then the corresponding phase factor in $\psi\,\mathrm{e}^{-\mathrm{i}\omega t}$ is, from (2.20), given by

$$\Theta = k_1x + k_2y - \omega t,$$

where

$$k_1 = (b/\sigma) + \nu\cos\beta, \quad k_2 = \nu\sin\beta. \tag{2.22}$$

On eliminating β and using (2.15), it follows that

$$\omega = 2bfk_1/(k_1^2+k_2^2+b^2) \equiv \omega(k_1,k_2). \tag{2.23}$$

The components of the group velocity are therefore (Lighthill 1965) given by

$$\frac{\partial\omega}{\partial k_1} = \frac{2bf(k_2^2+b^2-k_1^2)}{(k_1^2+k_2^2+b^2)^2}, \quad \frac{\partial\omega}{\partial k_2} = -\frac{4bfk_1k_2}{(k_1^2+k_2^2+b^2)^2}, \tag{2.24}$$

and substitution of (2.18), (2.19) shows that

$$\left(\frac{\partial\omega}{\partial k_1}, \frac{\partial\omega}{\partial k_2}\right) = -\frac{2\omega\nu}{(k_1^2+k_2^2+b^2)}(\cos\beta, \sin\beta).$$

Hence, in the coastal region $0 \leqslant y \leqslant l$, the group velocity at any point is directed exactly opposite to the local wavenumber in χ (phase velocity of $\chi e^{-i\omega t}$). This result provides the radiation condition to be applied in the subsequent sections. An equivalent procedure is to consider the time reversed when solving for χ.

The direction of the group velocity vector on the shelf is determined solely from condition (2.12), which corresponds to the semicircle shown in the wavenumber plane displayed in figure 1. The discrete eigenvalues are determined by the application of a further relation between α and k, depending on the parameters of the problem. If this were independent of k, as in the usual type of waveguide, then the repeated eigenvalues would have zero alongshore component of group velocity. However, since the prescribed condition (2.13) yields the curves shown in figure 1, a common tangent to the semicircle and a curve must be inclined to the k-axis, and hence the resonant mode must have a non-zero alongshore component of group velocity.

On considering again the shelf wave modes, given by (2.11), an interesting possibility arises when $\nu l = (\nu l)_N$, given by (2.18), for some N, as then $\delta_N = \frac{1}{2}\pi$ and the corresponding shelf wave mode has $k = b/\sigma$, $\alpha = \nu$. Then comparison of (2.11) and (2.20) shows that in this case $\chi(x, y)$ is a multiple of $\sin \nu y$ and therefore energy cannot propagate along the shelf in this critical mode but is trapped stationary in the coastal region. Though such a mode may be regarded as a feature of the mathematical mode, having little relevance to physical reality, the appearance of the $\sin \nu y$ solution in the shelf region plays a significant role in the subsequent analysis, in which $\nu l \neq (\nu l)_N$ for any N. If a disturbance on the shelf is regarded as the superposition of plane waves travelling in all directions, then, except along the offshore rays, energy is propagated away from the disturbance with reflexions at the shore and shelf–ocean boundary, and leakage into the ocean region. In contrast, energy is 'trapped' on the offshore rays whose influence on the structure of the solution is thereby enhanced.

The group velocity of a shelf wave in the ocean region has x-component $d\omega/dk = f d\sigma/dk$, where $\sigma \equiv \sigma[k, \alpha(k)]$ is determined from (2.12) and (2.13). After some manipulation, it is found that

$$\frac{1}{f}\frac{d\omega}{dk} = -\frac{\sigma^2}{k}\left(\frac{1}{\sigma}+1\right)\frac{k-b+2kl(k-b/\sigma)}{k+b+2bkl(1+1/\sigma)} \tag{2.25}$$

which, on comparison with (2.16) and (2.17), is positive or negative according as k/b is less or greater than $\hat{k}$. The modes, one of which is resonant, corresponding to a repeated eigenvalue have zero group velocity in the ocean but propagate energy slowly northwards in the coastal region. In contrast, a mode corresponding to $\delta_N = \frac{1}{2}\pi$, in which energy is trapped on the shelf, is such that energy in the ocean region propagates slowly southwards because, from (2.25),

$$(d\omega/dk)_{k=b/\sigma} = -\sigma^3(1-\sigma)f/b(2bl+\sigma). \tag{2.26}$$

Between these two a mode having $\hat{k} < k/b < \sigma^{-1}$ is such that energy is propagated northwards on the shelf but southwards in the ocean, very slowly in both regions.

For any mode, the alongshore component of group velocity in the shelf region is greater than the group velocity in the ocean region, their difference being

$$\sigma^2\alpha^2 f/bk[k+b+2bkl(1+1/\sigma)],$$

after use of (2.24) and (2.25). Since this quantity is small for all modes, there was some justification for its neglect by previous authors who considered only expression (2.25). However the above analysis gives a better understanding of the energy-trapping features of this mode as well as providing the exact radiation condition to be applied to scattering problems.

Since the shelf wave modes are such that the wavenumber has magnitude $(k^2+\alpha^2)^{\frac{1}{2}} = (2bk/\sigma - b^2)^{\frac{1}{2}}$ in $0 \leqslant y \leqslant l$ and k in $y \geqslant l$, it follows from the above calculations that, in both regions, shorter wavelengths occur for shelf waves propagating energy southwards than for those with positive x-component of the group velocity, the disparity being greatest for the first modes of each type and decreasing as α takes values closer to ν.

The problems to be discussed here concern the scattering by a fixed obstacle projecting from the shoreline of a (subcritical) shelf wave with lowest α-mode ($\frac{1}{2}\pi < \alpha l < \pi$) propagating energy northward. By rescaling (2.11), the incident wave ψ_i is defined by

$$\psi_i = \mathrm{e}^{\mathrm{i}kx+by} \sin \alpha y \quad (0 \leqslant y \leqslant l), \tag{2.27}$$

$$\psi_i = \mathrm{e}^{k(\mathrm{i}x-y)+(k+b)l} \sin \alpha l \quad (y \geqslant l), \tag{2.28}$$

where (with $\gamma_1 = \gamma$) $k = b/\sigma - \nu \cos \gamma$, $\alpha = \nu \sin \gamma$. The corresponding χ_i is, from (2.20), given by

$$\chi_i(x, y; \gamma) = \mathrm{e}^{-\mathrm{i}\nu x \cos \gamma} \sin(\nu y \sin \gamma) = (1/2\mathrm{i})\,[\mathrm{e}^{-\mathrm{i}\nu(x\cos\gamma - y \sin\gamma)} - \mathrm{e}^{-\mathrm{i}\nu(x \cos\gamma + y \sin\gamma)}]. \tag{2.29}$$

This expression shows that ψ_i, given by (2.27), corresponds, at each point, to a pair of waves with group velocities at angles to the positive x-axis and which together propagate energy solely northwards. These waves are repeatedly reflected at the shoreline $y = 0$ and also at $y = l$ in such a way as to support an edge wave, defined by (2.28), travelling northwards in the ocean region along the shelf boundary.

Let ψ (and the corresponding χ) measure the scattered waves due to the presence of the barrier. Then ψ and χ satisfy (2.7) and (2.21) respectively, while, on using (2.20), the boundary conditions corresponding to (2.8) and the continuity requirement (2.9) are

$$\chi(x, 0) = 0, \tag{2.30}$$

$$\left.\begin{aligned} \chi(x, l) &= \psi(x, l)\,\mathrm{e}^{-b(l+\mathrm{i}x/\sigma)} \\ \frac{\partial \chi}{\partial y}(x, l) &= \left[\frac{\partial \psi}{\partial y}(x, l) - b\psi(x, l)\right]\mathrm{e}^{-b(l+\mathrm{i}x/\sigma)}. \end{aligned}\right\} \tag{2.31}$$

The inhomogeneous condition is that the incident field must be cancelled on the obstacle, and the solution is made unique by applying the radiation condition.

Physical reality ensures that resonances of the type discussed above cannot occur over any significant length of coastline but must only appear as localized phenomena, as observed. The formulation of a scattering problem requires some averaging of the sea depth contours near the coastline which realistically must yield a value of νl such that

$$(\nu l)_N < \nu l < (\widehat{\nu l})_{N+1}$$

for some N, with νl not too close to either end value. Then there are N subcritical and N supercritical modes which, since $\gamma_m \neq \delta_n$ for all pairs (m, n), have distinct offshore wave-numbers on the shelf, and distinct exponential decay rates in the ocean. Consequently, the apparently simple mathematical problem of total reflexion by an infinite offshore barrier (along the y-axis) is seen to be anything but simple and currently unsolved. This is due to the difficulties encountered when the scatterer straddles the $y = l$ line separating the shelf and ocean regions. These problems are avoided in the present analysis which depends crucially on

The reflexion and transmission of a plane wave at the shelf boundary $y = l$ are obtained by considering the wave wavefunction χ' and the harmonic function ψ' defined by

$$\chi' = e^{-i\nu x\cos\tau}[e^{-i\nu y\sin\tau} - p(\tau)\,e^{-i\nu(2l-y)\sin\tau}] \quad (y \leqslant l), \tag{3.6}$$

$$\psi' = q(\tau)\exp\{i(b/\sigma - \nu\cos\tau)[x \pm i(y-l)] + bl\} \quad (y \geqslant l), \tag{3.7}$$

where the sign is chosen according to whether $(b/\sigma - \nu\cos\tau)$ has positive or negative real part in order that $\psi' \to 0$ as $y \to \infty$. On applying conditions (2.31), it follows that

$$\frac{1+p(\tau)}{1-p(\tau)} = \frac{\pm(b/\sigma - \nu\cos\tau)+b}{i\nu\sin\tau}, \quad q(\tau) = [1-p(\tau)]\,e^{-i\nu l\sin\tau}.$$

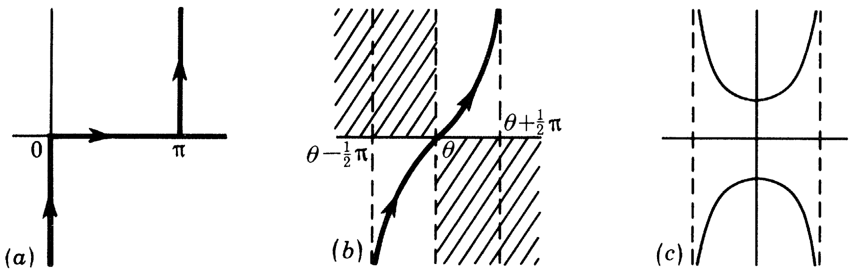

FIGURE 2. (a) The contour C; (b) The contour $S(\theta)$; (c) The curves $\cos\tau_1\cosh\tau_2 = \cosh c$ through $(0, \pm c)$.

At this stage it is convenient to define the real positive number c such that $b/\sigma = \nu\cosh c$, whence, from (2.15):

$$\cosh c = (1-\sigma^2)^{-\frac{1}{2}}, \quad \tanh c = \sigma, \quad \nu\sinh c = b. \tag{3.8}$$

Then

$$p(\tau) = (e^{\pm c} - e^{\pm i\tau})/(e^{\pm c} - e^{\mp i\tau}) \tag{3.9}$$

according as $\cos\tau_1\cosh\tau_2 \lessgtr \cosh c$. The equation $\cos\tau_1\cosh\tau_2 = \cosh c$ defines two curves, symmetrically placed either side of the τ_1-axis in the interval $|\tau_1 - 2n\pi| < \frac{1}{2}\pi$ for all integers n. Only the interval $|\tau_1| < \frac{1}{2}\pi$ is relevant here and it is illustrated in figure 2c. The construction (3.6), (3.7) remains valid at the point $\tau = -ic$ on C, where it reduces to

$$\chi' = e^{-b(y+ix/\sigma)}, \quad \psi' = 1.$$

Now condition (2.30) implies that the reflected wave in (3.6) will itself be reflected at the shoreline to yield an additional term

$$p(\tau)\,e^{-i\nu[x\cos\tau + (2l+y)\sin\tau]}.$$

Hence infinitely many reflexions occur and, from (3.5), the right-hand side of (3.2) is completely determined:

$$G_\chi(x, y; Y) = \tfrac{1}{4}iH_0^{(2)}(\nu R) - \tfrac{1}{4}iH_0^{(2)}(\nu R')$$
$$+\frac{i}{\pi}\int_C P(\tau)\,e^{-i\nu x\cos\tau}\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\,d\tau, \tag{3.10}$$

where, after a summation under the integral sign,

$$P(\tau) = p(\tau)\,[e^{2i\nu l\sin\tau} - p(\tau)]^{-1} \tag{3.11}$$

and $p(\tau)$ is defined by (3.9). Also

$$G_\psi(x, y; Y) = -\frac{1}{2\pi}\int_C Q(\tau)\exp\{\mathrm{i}(b/\sigma - \nu\cos\tau)[x \pm \mathrm{i}(y-l)] + bl\}\sin(\nu Y\sin\tau)\,\mathrm{d}\tau, \quad (3.12)$$

where

$$Q(\tau) = \mathrm{e}^{\mathrm{i}\nu l\sin\tau}[1-p(\tau)][\mathrm{e}^{2\mathrm{i}\nu l\sin\tau} - p(\tau)]^{-1}. \quad (3.13)$$

These results could obviously have been obtained directly, but with their physical significance less apparent, by writing $2\mathrm{i}P(\tau)\sin(\nu y\sin\tau)$ instead of $p(\tau)\,\mathrm{e}^{-\mathrm{i}\nu(2l-y)\sin\tau}$ in (3.6), and $Q(\tau)$ instead of $q(\tau)$ in (3.7).

The crucial feature that makes the above construction of G_χ and G_ψ possible, without meeting eigenvalue restrictions, is that, according to (3.5), the forcing terms in (3.2) can, for $y > Y$, be expressed solely in terms of waves propagating energy away from the shoreline, without violating the zero condition (2.30). This is achieved because the representation (3.4) is different according as $y-Y$ is positive or negative, with the result that if $0 \leqslant y < Y$, y and Y must be interchanged on the right-hand side of (3.5). The singular term $\frac{1}{4}\mathrm{i}H_0^{(2)}(\nu R)$ propagates energy towards or away from the shore according as y is less or greater than Y, but the image term $\frac{1}{4}\mathrm{i}H_0^{(2)}(\nu R')$, which corresponds to total reflexion at the shoreline, everywhere propagates energy away from the shore. The integration with respect to τ along C in (3.4) corresponds to the x-component of wavenumber, $\nu\cos\tau$, taking all real values from $-\infty$ to ∞, as in a Fourier transform. The portion of C on the real axis (figure 2a) has $|\cos\tau| < 1$, $\sin\tau$ real and positive, and yields the superposition of plane waves. The arms of C, $\tau_1 = 0$, $\tau_2 \leqslant 0$ and $\tau_1 = \pi$, $\tau_2 \geqslant 0$, have

$$\cos\tau = \pm\cosh\tau_2, \quad \mathrm{i}\sin\tau = \sinh|\tau_2|,$$

and hence correspond to the superpositions of shorter guided waves (wavenumber greater than ν) travelling alongshore in either direction with exponential decay either side of the line $y = Y$.

Now, from (3.9), (3.11) and (3.13), it is evident that $P(\tau)$, $Q(\tau)$ can become infinite on the real axis ($\tau = \tau_1$), where, for all τ_1, $|p(\tau_1)| = 1$ and

$$\arg p(\tau_1) = -2\arctan[\sin\tau_1/(\mathrm{e}^c - \cos\tau_1)],$$

which takes values between $\pm 2\arcsin(\mathrm{e}^{-c})$. The integrands of (3.10), (3.12) remain regular at 0, π but poles occur when

$$\nu l\sin\tau_1 + \arctan[\sin\tau_1/(\mathrm{e}^c - \cos\tau_1)] = m\pi \quad (m = 1, 2, \ldots),$$

which, with c given by (3.8), implies that

$$\nu\sin\tau_1\cot(\nu l\sin\tau_1) + b + b/\sigma - \nu\cos\tau_1 = 0.$$

Comparison with (2.12), (2.13) shows that these poles correspond to the possible shelf wave modes at the given σ and bl. The values of τ_1 in $(0, \frac{1}{2}\pi)$ are $\{\gamma_m; 1 \leqslant m \leqslant N\}$, which yield the subcritical modes, while those in $(\frac{1}{2}\pi, \pi)$ are $\{\pi - \delta_m; 1 \leqslant m \leqslant N\}$, which yield the supercritical modes.

Since, for complex values $\tau_1 + i\tau_2$ of τ, the function $e^{2i\nu l \sin\tau}$ has modulus

$$\exp(-2\nu l \cos\tau_1 \sinh\tau_2),$$

the complex zeros of $e^{2i\nu l\sin\tau} - p(\tau)$ must have $\cos\tau_1$ small, in which case the upper signs in (3.9) are appropriate. There are evidently infinitely many discrete zeros $\{\tau_n^+, \tau_n^-; n \geqslant 1\}$ asymptotic to the line $\tau_1 = \frac{1}{2}\pi$ such that Re $(\frac{1}{2}\pi - \tau_n^+)$, Im $\tau_n^+ > 0$ and Re $(\frac{1}{2}\pi - \tau_n^-)$, Im $\tau_n^- < 0$. These poles of $P(\tau)$ yield residues with the factor $e^{-i\nu x\cos\tau_n^+}$ or $e^{-i\nu x\cos\tau_n^-}$, which have rapid exponential decay in the positive or negative x-directions respectively. Thus the value of x must be taken into account when considering deformations of the contour C in (3.10) and (3.12).

The radiation condition requires that this contour C be indented below the poles $\{\gamma_m\}$ and above the poles $\{\pi - \delta_m\}$. Then deformation to $S(\frac{1}{2}\pi)$, denoted by S, shows that (3.10) can be rewritten

$$\begin{aligned} G_\chi(x, y; Y) &= \tfrac{1}{4}iH_0^{(2)}(\nu R) - \tfrac{1}{4}iH_0^{(2)}(\nu R') \\ &\quad + \frac{i}{\pi}\int_S P_+(\tau)\, e^{-i\nu x\cos\tau} \sin(\nu y \sin\tau)\sin(\nu Y\sin\tau)\, d\tau \\ &\quad + \frac{i}{\pi}\int_{-i\infty}^{-ic} [P_-(\tau) - P_+(\tau)]\, e^{-i\nu x\cos\tau}\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\, d\tau, \end{aligned} \tag{3.14}$$

where the analytic functions P_+, P_- are obtained by taking the upper or lower signs respectively in (3.9). Now, in an obvious notation,

$$P_-(\tau) - P_+(\tau) = \frac{e^{-2i\nu l\sin\tau}[p_-(\tau) - p_+(\tau)]}{1 - [p_-(\tau) + p_+(\tau)]\, e^{-2i\nu l\sin\tau} + p_-(\tau)p_+(\tau)\, e^{-4i\nu l\sin\tau}},$$

where, from (3.9), the denominator reduces to 1 at $\tau = -ic$, and

$$p_-(\tau) - p_+(\tau) = 2i\sin\tau(\cosh c - \cos\tau)/[1 - \cosh(c + i\tau)].$$

Hence, by application of Watson's lemma,

$$\begin{aligned} &\int_{-i\infty}^{-ic} [P_-(\tau) - P_+(\tau)]\, e^{-i\nu x\cos\tau}\, d\tau \\ &\quad\sim \int_{-i\infty}^{-ic} \frac{2i\sin\tau(\cosh c - \cos\tau)}{1 - \cosh(c + i\tau)}\, e^{-i\nu(x\cos\tau + 2l\sin\tau)}\, d\tau \\ &\quad\sim \frac{i\, e^{-i\nu(x\cosh c - 2il\sinh c)}}{\nu^2(ix\sinh c + 2l\cosh c)^2} \\ &\quad= \frac{i\, e^{-b(2l + ix/\sigma)}}{b^2(ix + 2l/\sigma)^2}, \end{aligned}$$

after substitution of (3.8). Thus the last integral in (3.14) is given asymptotically by

$$\begin{aligned} &\frac{i}{\pi}\int_{-i\infty}^{-ic} [P_-(\tau) - P_+(\tau)]\, e^{-i\nu x\cos\tau}\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\, d\tau \\ &\quad\sim \frac{\sigma^2 e^{-ibx/\sigma}}{4\pi b^2}\left\{\frac{e^{-b(2l-y-Y)}}{(i\sigma x + 2l - y - Y)^2} - \frac{e^{-b(2l-y+Y)}}{(i\sigma x + 2l - y + Y)^2} - \frac{e^{-b(2l+y-Y)}}{(i\sigma x + 2l + y - Y)^2}\right. \\ &\qquad\left. + \frac{e^{-b(2l+y+Y)}}{(i\sigma x + 2l + y + Y)^2}\right\}, \end{aligned} \tag{3.15}$$

This result corresponds to those obtained by Buchwald (1977) and Chao *et al.* (1979), who too obstacles small enough for a Fourier transform in x to be applied to the scattering problem After solving ordinary differential equations in y, the inverse transform was estimated at larg distance by deforming the path of integration to a large semi-circle at infinity and neglectin the branch cut integrals. The remark by Chao *et al.* that scattering favours higher mode corresponds to the denominators $E(\gamma_m)$, $E(\pi-\delta_m)$, where $E(\tau)$ is defined by (3.19), takin smaller values as γ_m, δ_m increase. Note that $E(\tau)$ vanishes when $k(\tau)$ satisfies (2.16), i.e. a resonance.

The validity of expansions (3.20a, b) depends on the earlier assumption that $(\nu l)_N < \nu l <$ $(\widehat{\nu l})_{N+1}$ for some N, which eliminates the following difficulties.

If $\nu l = (\widehat{\nu l})_{N+1}$, then comparison of (3.19) with (2.16) shows that $P(\tau)$ has a double pole corresponding to a repeated eigenvalue whose modes, according to (2.25), have zero grou velocity in the ocean region. Then expression (3.20a) would be augmented by the residue a this double pole, which is a linear combination of two modes, one of which is resonant.

If $\nu l = (\nu l)_N$, then $\delta_N = \frac{1}{2}\pi$, and a pole of $P(\tau)$ lies on the path of integration S. Then Cauchy principal value of the integral would be required, to satisfy the radiation condition with the result that the Nth term in the first sum of (3.20b), namely $-2\mathrm{i}[E(\frac{1}{2}\pi)]^{-1}\sin(\nu y)\sin(\nu Y)$ would be halved and a corresponding term added to (3.20a).

If $(\widehat{\nu l})_N < \nu l < (\nu l)_N$, then the Nth term of (3.20b) would be subcritical and therefor appear instead in (3.20a), with both this and the existing Nth subcritical mode having ver small denominators.

However, in the near field, expansions (3.20a, b) are inappropriate and it is necessary to retain the source and image source terms apart from the integral in (3.16). Using the relation

$$P(\tau) = p(\tau)\,\mathrm{e}^{-2\mathrm{i}\nu l\sin\tau}[1+P(\tau)],$$

derived from (3.11), it follows by repeated application that

$$\begin{aligned}\frac{\mathrm{i}}{\pi}\int_S P_+(\tau)\,\mathrm{e}^{-\mathrm{i}\nu x\cos\tau}\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\,\mathrm{d}\tau \\ =\frac{\mathrm{i}}{\pi}\sum_{n=1}^{M}\int_S[p_+(\tau)]^n\,\mathrm{e}^{-\mathrm{i}\nu(2nl\sin\tau+x\cos\tau)}\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\,\mathrm{d}\tau \\ +\frac{\mathrm{i}}{\pi}\int_S[p_+(\tau)]^M\,\mathrm{e}^{-\mathrm{i}\nu(2Ml\sin\tau+x\cos\tau)}P_+(\tau)\sin(\nu y\sin\tau)\sin(\nu Y\sin\tau)\,\mathrm{d}\tau. \quad (3.21\end{aligned}$$

Since $p_+(\tau)$ is a slowly varying function, the series consists of integrals of the type (3.4), wit the saddle points moving towards $\frac{1}{2}\pi$ as n increases. The nth integral yields contributions t G_χ corresponding to rays reflected n times at $y = l$, and n or $n\pm1$ times at $y = 0$, accordin to the exponential components of the sine functions. The evaluation of $p_+(\tau)$ at a saddle poin corresponds to taking the appropriate reflexion coefficient for each such ray. The expansio described by (3.21) can be continued indefinitely because the assumption that $\tau = \frac{1}{2}\pi$ is no a pole of $P_+(\tau)$ implies that the residual integral tends to zero as $M\to\infty$. This assumptio also ensures that, after a finite number of steps, all subsequent saddle points are nearer t $\tau = \frac{1}{2}\pi$ than the poles of $P_+(\tau)$. This ability to move the saddle points relative to the poles i due to the latter occurring when the rapidly oscillating function $\mathrm{e}^{2\mathrm{i}\nu l\sin\tau}$ equals the slowl varying $p_+(\tau)$. It should be contrasted with the canonical form of a plane wave representatio

The above analysis depends crucially on there being no pole at $\tau = \frac{1}{2}\pi$, i.e. no shelf wave mode in which energy is trapped on the shelf. The exclusion of the double pole is less important here because the source lies on the shelf, forcing the radiation condition to be applied in this region.

Turning attention now to G_ψ, a slightly different approach is required when deforming the contour in (3.12) because the change to an analytic integrand involves the expression

$$Q_+(\tau)\,e^{-(b/\sigma-\nu\cos\tau)(y-l)} - Q_-(\tau)\,e^{(b/\sigma-\nu\cos\tau)(y-l)}$$

which, for large enough y, becomes exponentially large as $\tau \to -i\infty$. This difficulty is overcome by arranging to integrate along a curve on which $b/\sigma - \nu\cos\tau$ is pure imaginary. For $x \neq 0$, let the contour C in (3.12) be deformed, according as x is positive or negative, to $S(0)$ or $S(\pi)$, respectively, along which the integral vanishes because (3.9), (3.13) imply that $Q(-\tau) = Q(\tau) = Q(\tau+2\pi)$ for all τ. The non-analyticity of the integrand leads to the branch cut integral

$$\frac{1}{2\pi}\int_{-i\infty-\frac{1}{2}\pi\,\mathrm{sgn}\,x}^{-ic}[Q_+(\tau)\,e^{-(b/\sigma-\nu\cos\tau)(y-l)} - Q_-(\tau)\,e^{(b/\sigma-\nu\cos\tau)(y-l)}]\,e^{i(b/\sigma-\nu\cos\tau)x+bl}\sin(\nu Y\sin\tau)\,d\tau$$

which is taken along one half of the lower curve in figure 2c, but unlike the last integral in (3.14), a large enough value of $b(l-y)$ does not ensure that this integral is negligibly small throughout the ocean region. It is so in the far field, yielding expansions in terms of shelf wave modes which, as expected, correspond exactly to those in (3.20a, b). Thus as $x \to \infty$,

$$G_\psi(x, y;\, Y) \sim 2i\sum_{m=1}^{N}\frac{\exp\{i(b/\sigma-\nu\cos\gamma_m)[x+i(y-l)]+bl\}}{E(\gamma_m)}\sin(\nu l\sin\gamma_m)\sin(\nu Y\sin\gamma_m) \tag{3.25a}$$

while, as $x \to -\infty$,

$$G_\psi(x, y;\, Y) \sim -2i\sum_{m=1}^{N}\frac{\exp\{i(b/\sigma+\nu\cos\delta_m)[x+i(y-l)]+bl\}}{E(\pi-\delta_m)}\sin(\nu l\sin\delta_m)\sin(\nu Y\sin\delta_m). \tag{3.25b}$$

Consideration of G_ψ in the near field is complicated and, since the details are not required in the subsequent analysis, they are not included here.

4. Numerical values

The physical phenomenon that motivates the present problem is the Brooks Peninsula on the west coast of Vancouver Island. Depth measurements in the vicinity of this peninsula of length $a = 20$ km show that the profile (2.5) is a good approximation when $H = 2$ km, $l = 50$ km and $h(a) = 100$ m. Then $bl = 2.496$, $h(0) = 13.57$ m. Now the dominant shelf wave mode is observed (Cutchin & Smith 1973) to have alongshore wavelength of approximately 1000 km, i.e. $kl \approx \frac{1}{10}\pi \approx 0.314$, $k/b = 0.126$. Then the lowest mode solution of (2.13) yields $\alpha/b \approx 0.973$, and subsequent substitution into (2.12) gives $\omega/f = \sigma \approx 0.128$. Since, at latitude 50°, the Coriolis parameter $f \approx 9.62$ rad/day, this value of σ corresponds to a period of 5 days, in agreement with observations. Further numerical values obtained are

$$\sigma^{-1} \approx 7.800, \quad \nu/b \approx 7.736, \quad \nu l \approx 19.31, \quad \nu a \approx 7.724,$$

so the use of short wave asymptotics is appropriate. Resonant modes do not occur because $(\nu l)_6 < \nu l < (\widehat{\nu l})_7$ where, from the formulae (2.18) and (2.19), $(\nu l)_6 = 18.128$ and $(\widehat{\nu l})_7 \approx 21.265$. Thus, for the above numerical values of bl and σ, there are six subcritical and six supercritical shelf wave modes defined by angles $\{\gamma_m, \delta_m; 1 \leqslant m \leqslant 6\}$ which, with the corresponding wave-number components, are listed in table 1.

Table 1

m	γ_m	αl	kl	δ_m	αl	kl
1	7° 14′	2.429	0.314	9° 8′	3.067	38.53
2	15° 53′	5.283	0.898	18° 31′	6.132	37.78
3	25° 38′	8.353	2.061	28° 26′	9.193	36.45
4	36° 35′	11.51	3.964	39° 22′	12.25	34.40
5	49° 37′	14.71	6.960	52° 20′	15.28	31.27
6	68° 33′	17.97	12.41	71° 10′	18.27	25.70

As mentioned in §2, the shelf waves propagating energy southwards have shorter wavelengths than those with positive component of group velocity, the disparity being greatest for the first modes of each type. The lowest offshore supercritical mode has x-component of wavenumber equal to $\nu \cos \delta_1 + b\sigma^{-1}$, which differs little from $\nu \cos \gamma + b\sigma^{-1}$. Its value is $15.438b$, which corresponds to an alongshore wavelength of approximately 8 km, about 1/120 that for the lowest offshore subcritical mode.

With the above values of σ, bl and f, the group velocity in (2.26) is about 1.7 km/day, and a similar speed is found for the northward rate of energy propagation on the shelf when the group velocity in the ocean is zero. Thus although energy is not trapped simultaneously in the two regions, according to the present mathematical model, the disparity would be difficult to detect in practice.

If the source introduced in §3 is placed on the barrier, i.e. $0 < Y \leqslant a$, then insertion of the above values of σ, bl, a/l in (3.15) shows that the largest term, the first, has magnitude bounded by

$$\sigma^2 e^{-b(l-a)}/4\pi b^2 (l-a)^2 = 1.300 \times 10^{-4}.$$

5. Method of solution

For the offshore barrier, equations (2.7), (2.21) and conditions (2.10), (2.30), (2.31) are satisfied by the representations

$$\begin{Bmatrix} \chi(x, y) \\ \psi(x, y) \end{Bmatrix} = \int_0^a \mu(Y) \begin{Bmatrix} G_\chi(x, y; Y) \\ G_\psi(x, y; Y) \end{Bmatrix} dY \tag{5.1}$$

in which equation (3.2) ensures that

$$\mu(y) = [\partial\chi/\partial x]_{x=0-}^{x=0+} \quad (0 < y < a). \tag{5.2}$$

Thus the scattered field may be regarded as being due to a distribution of sources on the barrier with density function $\mu(y)$ related to the normal derivative discontinuity in χ, i.e. discontinuity in the offshore velocity component.

Since condition (2.32) takes the form

$$\chi(0, y) = -\sin(\nu y \sin \gamma) \quad (0 < y < a), \tag{5.3}$$

the representation (5.1) must be such that

$$\int_0^a \mu(Y) G_\chi(0, y; Y) \, dY = -\sin(\nu y \sin \gamma) \quad (0 < y < a), \tag{5.4}$$

which determines $\mu(y)$. Thus the problem is equivalent to that of solving an integral equation of the first kind with symmetric kernel, whose values are difficult to compute accurately. This is overcome in the solution presented below by constructing an iterative process that exploits known methods and solutions in the limit $bl \to \infty$ and the large value (*ca.* 23.2) of $2\nu(l-a)$ which shows that energy scattered offshore and reflected back to the barrier must travel some four wavelengths. The procedure is more clearly explained by considering not just the offshore barrier but the more general case of an obstacle bounded by the curve Γ defined in §2. The representation of χ as the field due to a source distribution on Γ of density $\mu(s)$ is evidently of the form

$$\chi(X, Y) = \int_\Gamma \mu(s)\, G_\chi(X-x, Y; y)\, \mathrm{d}s. \tag{5.5}$$

The corresponding disturbance in the ocean region is given by

$$\psi(X, Y) = \int_\Gamma \mu(s)\, G_\psi(X-x, Y; y)\, \mathrm{d}s$$

On letting $(X, Y) \to \Gamma$, the application of condition (2.32) yields

$$\int_\Gamma \mu(s;\gamma)\, [G_\chi(X-x, Y; y)]_{(X,Y)\in\Gamma}\, \mathrm{d}s = -[\chi_i(X, Y; \gamma)]_\Gamma. \tag{5.6}$$

Now, in the $bl \to \infty$ limit, the scattering problem may be regarded as that due to the incidence of the two plane wave components of χ_i, given by (2.29), on the closed body D formed by the curve Γ and its image Γ' in the x-axis, with no other boundaries present. Since this problem has received much attention in the literature, the required source density $\mu_0(s;\gamma)$ in this limit will be regarded as a known function. Thus, in (5.6)

$$\int_\Gamma \mu_0(s;\gamma)\, [G^0_\chi(X-x, Y; y)]_{(X,Y)\in\Gamma}\, \mathrm{d}s = -[\chi_i(X, Y; \gamma)]_\Gamma, \tag{5.7}$$

where $G^0_x = \lim_{bl\to\infty} G_\chi$ and from (3.10) is given by

$$G^0_\chi(X-x, Y; y) = \tfrac{1}{4}\mathrm{i}H^{(2)}_0(\nu\rho) - \tfrac{1}{4}\mathrm{i}H^{(2)}_0(\nu\rho'), \tag{5.8}$$

where

$$\rho = [(x-X)^2 + (y-Y)^2]^{\frac{1}{2}}, \quad \rho' = [(x-X)^2 + (y+Y)^2]^{\frac{1}{2}}. \tag{5.9}$$

On writing

$$\mu(s;\gamma) = \mu_0(s;\gamma) + \mu^*(s;\gamma) \tag{5.10}$$
$$G_\chi(X-x, Y; y) = G^0_\chi(X-x, Y; y) + G^*_\chi(X-x, Y; y),$$

it follows, on using (5.7), that the integral equation (5.6) can be written in the form

$$\int_\Gamma \mu^*(s;\gamma)\, [G^0_\chi(X-x, Y; y)]_{(X,Y)\in\Gamma}\, \mathrm{d}s = -\int_\Gamma \mu(s;\gamma)\, [G^*_\chi(X-x, Y; y)]_{(X,Y)\in\Gamma}\, \mathrm{d}s. \tag{5.11}$$

Now (3.10) and (2.29) imply that

$$G^*_\chi(X-x, Y; y) = \frac{\mathrm{i}}{\pi}\int_C P(\tau)\, \chi_i(X, Y; \tau)\, \mathrm{e}^{\mathrm{i}\nu x\cos\tau} \sin(\nu y \sin\tau)\, \mathrm{d}\tau, \tag{5.12}$$

so the unknown right-hand side of (5.11) can be regarded as the superposition of terms like the right-hand side of (5.7), with the angle ranging over all values on the contour C. The scattering problem in the $bl \to \infty$ limit is well posed for a pair of plane waves incident at any angle τ in the real interval $(0, \pi)$, and the required source density $\mu_0(s;\tau)$ on Γ can be extended to complex τ by analytic continuation. Since the function $P(\tau)$ in (5.12) corresponds to multiple reflexions at $y = 0$ and $y = l$, this plane wave representation of G^*_χ shows that, at any point (X, Y), all reflexions must occur in pairs of the type $\chi_i(X, Y; \tau)$. By exploiting this analogy and the extension of (5.7) to all angles of incidence, it follows on substitution in (5.11) that the source density $\mu(s;\gamma)$ must satisfy

$$\mu^*(s;\gamma) = \mu(s;\gamma) - \mu_0(s;\gamma)$$
$$= \frac{\mathrm{i}}{\pi}\int_\Gamma \mu(s';\gamma)\int_C P(\tau)\,[\mathrm{e}^{\mathrm{i}\nu x\cos\tau}\sin(\nu y\sin\tau)]_{(x,y)\in\Gamma}\,\mu_0(s;\tau)\,\mathrm{d}\tau\,\mathrm{d}s'. \quad (5.13)$$

On writing

$$K(s,s') = \frac{\mathrm{i}}{\pi}\int_C P(\tau)\,[\mathrm{e}^{\mathrm{i}\nu x\cos\tau}\sin(\nu y\sin\tau)]_{(x,y)\in\Gamma}\,\mu_0(s;\tau)\,\mathrm{d}\tau, \quad (5.14)$$

where $[\;]_\Gamma = [\chi_i(x,y;\pi-\tau)]_\Gamma$ is expressed as a function of s', the integral equation of the second kind for $\mu(s;\gamma)$ on Γ is seen to have a kernel $K(s,s')$ independent of γ and whose dependence on the shape of Γ is expressed solely by the function $\mu_0(s;\tau)$, i.e. knowledge of scattering by D of all possible incident plane wave pairs. The dependence of $\mu(s;\gamma)$ on γ arises only from the forcing function $\mu_0(s;\gamma)$. Equation (5.13) can also be written as an integral equation for $\mu^*(s;\gamma)$, namely

$$\mu^*(s;\gamma) - \mu_1(s;\gamma) = \int_\Gamma \mu^*(s';\gamma)\,K(s,s')\,\mathrm{d}s', \quad (5.15)$$

where

$$\mu_1(s;\gamma) = \int_\Gamma \mu_0(s';\gamma)\,K(s,s')\,\mathrm{d}s', \quad (5.16)$$

and K is defined by (5.14). Thus, by subtracting out the solution in the limit $bl \to \infty$, the integral equation of the first kind (5.6) is replaced by one of the second kind (5.15). This formulation is advantageous if K is a small kernel, which seems likely because the rapid variation of $P(\tau)$ cannot be cancelled by the other terms in the integrand of (5.14). With the scattering obstacle of limited extent, the dominant contributions to this integral are from values of τ near $\frac{1}{2}\pi$ and are estimated by deforming the contour to steepest descent paths as done for the near field of G_χ in §3.

The functions $\mu_1(s;\gamma)$, $\mu_2(s;\gamma)$ etc., obtained by solving iteratively the integral equation (5.15), are those found from the following iterative procedure for solving (5.11):

$$\int_\Gamma \mu_n(s;\gamma)\,[G^0_\chi(X-x, Y;y)]_{(X,Y)\in\Gamma}\,\mathrm{d}s = -\int_\Gamma \mu_{n-1}(s;\gamma)\,[G^*_\chi(X-x,Y;y)]_{(X,Y)\in\Gamma}\,\mathrm{d}s \quad (n \geqslant 1).$$

Thus the nth iteration considers the scattering by the obstacle in isolation of the superposition of pairs of plane waves due to reflexions at $y = 0$, l of the field scattered at the $(n-1)$th iteration. Physically, the obstacle is sufficiently far in terms of wavelengths from the edge of the shelf for a convergent method of solution to be obtained by alternately considering one and ignoring the other, with the shoreline condition (2.30) maintained throughout.

Mathematically, the solution proceeds by solving the basic scattering problem, represented by (5.7), and subsequently evaluating the kernel $K(s,s')$, given by (5.14), which is expected to be small enough for just a few terms of the iterative sequence $\{\mu_n(s;\gamma); n \geqslant 1\}$ of source density functions to provide a sufficiently accurate approximation to $\mu^*(s;\gamma) = \sum_{n=1}^{\infty}\mu_n(s;\gamma)$ and the corresponding field (5.5).

Exact solutions of (5.7), for suitably chosen Γ, are in terms of series involving special functions but these are slowly convergent. Their estimation is achieved by using a Watson transformation or similar reformulation, as illustrated in the next section which considers the simplest of such cases, the semicircular obstacle.

If the obstacle is slender, then an approximate solution of the basic scattering problem can be obtained either by transforming condition (2.32) to an approximate one on the shoreline or by using slender body theory in which the scattered field is represented as that due to singularities placed on the x-axis within the obstacle.

If the curve Γ is smooth and the closed body D is convex, then the methods of geometrical optics (first-order W.K.B. approximation) furnish a good approximation to the basic scattering problem. On substituting (5.8), equation (5.7) can be written

$$-[\chi_i(X, Y; \gamma)]_\Gamma = \tfrac{1}{4}\mathrm{i}\int_\Gamma \mu_0(s;\gamma)\,[H_0^{(2)}(\nu\rho)-H_0^{(2)}(\nu\rho')]\,\mathrm{d}s = \tfrac{1}{4}\mathrm{i}\int_{\Gamma+\Gamma'}\mu_0(s;\gamma)\,H_0^{(2)}(\nu\rho)\,\mathrm{d}s, \tag{5.17}$$

since the definition (2.29) of χ_i implies that

$$[\mu_0(s;\gamma)]_\Gamma = -[\mu_0(s;-\gamma)]_\Gamma = -[\mu_0(s;\gamma)]_{\Gamma'}.$$

On writing

$$\mu_0(s;\gamma) = \hat{\mu}_0(s;\gamma)-\hat{\mu}_0(s;-\gamma), \tag{5.18}$$

where

$$[\hat{\mu}_0(s;\pm\gamma)]_\Gamma = [\hat{\mu}_0(s;\mp\gamma)]_{\Gamma'},$$

equation (5.17) splits into the pair

$$-\hat{\chi}_i(X, Y;\pm\gamma) = -\frac{1}{2\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}\nu(X\cos\gamma\mp Y\sin\gamma)} = \tfrac{1}{4}\mathrm{i}\int_{\Gamma+\Gamma'}\hat{\mu}_0(s;\pm\gamma)\,H_0^{(2)}(\nu\rho)\,\mathrm{d}s, \tag{5.19}$$

i.e. the components $\hat{\mu}_0(s;\pm\gamma)$ of the source density correspond to the components $\hat{\chi}_i(X, Y;\pm\gamma)$ of χ_i. Now $\frac{1}{4}\mathrm{i}H_0^{(2)}(\nu\rho)$ is the free space Green function, and the Kirchhoff approximation states that, except in the penumbra region,

$$\hat{\mu}_0(s;\gamma) \sim 2\,\partial\hat{\chi}_i(x, y;\gamma)/\partial n \tag{5.20a}$$

on the portion of $(\Gamma+\Gamma')$ 'lit' by the incident plane wave $\hat{\chi}_i(x, y;\gamma)$, while

$$\hat{\mu}_0(s;\gamma) \sim 0 \tag{5.20b}$$

on the 'dark' portion of $(\Gamma+\Gamma')$. On Γ, there is a portion 'lit' directly by $\hat{\chi}_i(x, y;\gamma)$ and a smaller, possibly zero, portion lit by $\hat{\chi}_i(x, y;-\gamma)$, which corresponds to reflexions from the shoreline. The assumption of a convex scatterer removes the difficulties of multiple reflexions.

If the scattering body has a sharp corner, then from it radiate, in general, edge-defracted waves which are not predicted by geometrical optics. This is immaterial for the corners at each end of the double obstacle D when Γ does not meet the x-axis at right angles. However, it is a significant difficulty for the end of the offshore barrier whose consideration is the principal aim of this work. For this case, the construction of an approximate solution of the basic scattering problem requires the use of the complex Fresnel integral which takes account of edge diffracted waves and the lit and 'shadow' regions. A compensation here is that, after having found a sufficiently good approximation to $\mu_0(s;\tau)$, the subsequent evaluation of $K(s, s')$ is simplified by having $x = 0$ everywhere on Γ, the barrier.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

6. The semicircular obstacle

(a) *Determination of μ_0 in a suitable form*

On introducing polar coordinates by writing $x = \lambda\cos\phi$, $y = \lambda\sin\phi$, $X = \Lambda\cos\phi'$, $Y = \Lambda\sin\phi'$, a suitable choice of origin ensures that the boundary Γ for a semicircular obstacle of radius a is given by $\lambda = a$, $0 \leqslant \phi \leqslant \pi$. Then the expansions

$$e^{-i\nu x} = e^{-i\nu\lambda\cos\phi} = J_0(\nu\lambda) + 2\sum_{n=1}^{\infty} J_n(\nu\lambda)(-i)^n \cos n\phi$$

$$H_0^{(2)}(\nu\rho) = H_0^{(2)}\{\nu[\lambda^2+\Lambda^2-2\lambda\Lambda\cos(\phi-\phi')]^{\frac{1}{2}}\}$$
$$= J_0(\nu\lambda)H_0^{(2)}(\nu\Lambda) + 2\sum_{n=1}^{\infty} J_n(\nu\lambda)H_n^{(2)}(\nu\Lambda)\cos n(\phi-\phi') \quad (\Lambda \geqslant \lambda)$$

enable the incident field χ_i and limit Green function G_χ°, given by (2.29) and (5.8) respectively, to be written in the forms

$$\chi_i(x, y; \gamma) = 2i\sum_{n=1}^{\infty} J_n(\nu\lambda)(-i)^n \sin n\gamma \sin n\phi, \tag{6.1}$$

$$G_\chi^\circ(X-x, Y; y) = i\sum_{n=1}^{\infty} J_n(\nu\lambda)H_n^{(2)}(\nu\Lambda)\sin n\phi' \sin n\phi \quad (\Lambda \geqslant \lambda).$$

Then substitution in (5.7) yields

$$\mu_0(a\phi, \gamma) = -\frac{4}{\pi a}\sum_{n=1}^{\infty}\frac{(-i)^n}{H_n^{(2)}(\nu a)}\sin n\gamma \sin n\phi \tag{6.2}$$

but non-uniquely if $J_n(\nu a) = 0$ for some n. An alternative procedure, which yields (6.2) uniquely, is to require the source distribution to be such as to annihilate the incident field within the scattering obstacle. The scattered field in this $bl \to \infty$ limit is given, from (5.5), by

$$\hat{\chi}(X, Y; \gamma) = \int_0^\pi \mu_0(a\phi; \gamma)G_\chi^\circ(X - a\cos\phi, Y; a\sin\phi)\,a\,d\phi$$
$$= 2\sum_{n=1}^{\infty}(-i)^{n+1}\frac{J_n(\nu a)}{H_n^{(2)}(\nu a)}H_n^{(2)}(\nu\Lambda)\sin n\gamma \sin n\phi' \tag{6.3}$$

which is a sum of fields due to singularities of all orders placed at the origin.

Since $H_n^{(2)}(\nu a)$ cannot vanish, the expression (6.2) is defined for all νa. Indeed, here is an elementary example of how the null field method (Bates & Wall 1977) can eliminate the difficulties associated with irregular values of ν, described by Jones (1974) and Ursell (1973, 1978). The numerical values in §4, e.g. $\nu a \approx 8$, suggest that the series in (6.2) and (6.3) are slowly convergent and must be estimated by other means. One possibility is a Watson transformation in which each series is regarded as a sum of residues, and a corresponding contour integral is formed (Jones 1964, §8.7). The series in (6.3) is evaluated in this reference but the result does not obviously yield (6.10*a*, *b*), which therefore must be obtained from (6.2). Preferred here, and with some details included for explanatory purposes, is an alternative transformation, based on Poisson's sum formula:

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda)\,e^{2\pi i m\lambda}\,d\lambda,$$

used by Nussenzveig (1965), who considered in great detail the high frequency scattering by a sphere.

With his methods, consider

$$\kappa(\xi, \theta) = \sum_{n=1}^{\infty} \epsilon_n \mathrm{i}^n \frac{\cos n\theta}{H_n^{(1)}(\xi)}, \tag{6.4}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ $(n \geqslant 1)$ is Neumann's symbol. The relevance of this to (6.2) is that

$$\hat{\mu}_0(a\phi; \gamma) = (1/\pi a)\, \bar{\kappa}(\nu a, \phi+\gamma) \tag{6.5}$$

where $\hat{\mu}_0$ is defined by (5.18). The complex conjugate series is considered for closer analogy with Nussenzveig's analysis. The result of applying Poisson's sum formula to (6.4) is

$$\kappa(\xi, \theta) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{i}^n \mathrm{e}^{\mathrm{i}n\theta}}{H_n^{(1)}(\xi)} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda(\theta+\frac{1}{2}\pi)}}{H_\lambda^{(1)}(\xi)} \mathrm{e}^{2\pi \mathrm{i} m\lambda}\, \mathrm{d}\lambda,$$

and subsequent rearrangement with the use of the identity

$$H_{-\lambda}^{(1)}(\xi) = \mathrm{e}^{\mathrm{i}\pi\lambda} H_\lambda^{(1)}(\xi)$$

yields

$$\kappa(\xi, \theta) = 2\sum_{m=0}^{\infty} \frac{\cos \lambda\theta}{H_\lambda^{(1)}(\xi)} \mathrm{e}^{(2m+\frac{1}{2})\mathrm{i}\pi\lambda}\, \mathrm{d}\lambda. \tag{6.6}$$

Examination of the integrand as $|\lambda| \to \infty$ in the upper half of the complex λ-plane shows that, for all $m \geqslant 0$ and $|\theta| \leqslant \pi$, the path of integration can be closed by a large semicircle at infinity. The poles of the integrand occur at the zeros of $H_\lambda^{(1)}(\xi)$ in the upper half-plane which lie on a curve $h_1(\xi)$ in the first quadrant. This procedure yields a residue series consisting of exponentially decaying terms, provided $|\theta| < \frac{1}{2}\pi$. Thus $\kappa(\xi, \theta)$ is exponentially small on the dark half of the circle, and the physical significance of this diffracted field is described by Keller's geometrical theory of diffraction (Levy & Keller 1959). For $\frac{1}{2}\pi < \theta \leqslant \pi$, it is necessary to use the identity

$$\cos \lambda\theta = \mathrm{e}^{\mathrm{i}\pi\lambda} \cos \lambda(\pi-\theta) - \mathrm{i}\,\mathrm{e}^{\mathrm{i}\lambda(\pi-\theta)} \sin \lambda\pi$$

to rewrite (6.6) in the form

$$\kappa(\xi, \theta) = 2\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{\cos \lambda(\pi-\theta)}{H_\lambda^{(1)}(\xi)} \mathrm{e}^{(2m-\frac{1}{2})\mathrm{i}\pi\lambda}\, \mathrm{d}\lambda - \int_{-\infty+\mathrm{i}\epsilon}^{\infty+\mathrm{i}\epsilon} \frac{\mathrm{e}^{\mathrm{i}\lambda(\frac{1}{2}\pi-\theta)}}{H_\lambda^{(1)}(\xi)}\, \mathrm{d}\lambda, \tag{6.7}$$

where $\epsilon > 0$ and the m-summation in the second integral is valid wherever $\operatorname{Im} \lambda > 0$. The series of integrals in (6.7) yields exponentially small residues as above while the additional integral is estimated by deforming the path of integration to pass through the saddle point $\lambda = \xi \cos(\theta - \frac{1}{2}\pi)$ without crossing the curve $h_1(\xi)$. This saddle point is located by using the Debye asymptotic expansion for $H_\lambda^{(1)}(\xi)$, namely

$$H_\lambda^{(1)}(\xi) \sim (2/\pi)^{\frac{1}{2}} (\xi^2-\lambda^2)^{-\frac{1}{4}} \exp\{\mathrm{i}[(\xi^2-\lambda^2)^{\frac{1}{2}} - \lambda \arccos(\lambda/\xi) - \tfrac{1}{4}\pi]\}, \tag{6.8}$$

which is valid in a neighbourhood of the segment $-\xi < \lambda < \xi$ of the real λ-axis, with $(\xi^2-\lambda^2)^{-\frac{1}{4}} > 0$, $0 < \arccos(\lambda/\xi) < \pi$ on this segment. It then follows that, on the lit half of the circle $(\frac{1}{2}\pi < |\theta| \leqslant \pi)$, (6.7) yields

$$\kappa(\xi, \theta) \sim -\pi\xi \cos\theta\, \mathrm{e}^{\mathrm{i}\xi\cos\theta}.$$

The representation (6.6) ensures that $\kappa(\xi, \theta)$ has a smooth transition through the penumbra region close to $|\theta| = \frac{1}{2}\pi$. When these results for $\kappa(\xi, \theta)$ are substituted in (6.5), it follows that on Γ

$$\left.\begin{aligned} \hat{\mu}_0(a\phi;\gamma) &\sim 0 \quad (0 \leqslant \phi < \tfrac{1}{2}\pi-\gamma), \\ \hat{\mu}_0(a\phi;\gamma) &\sim -\nu\cos(\phi+\gamma)\,\mathrm{e}^{-\mathrm{i}\nu a\cos(\phi+\gamma)}\,(\tfrac{1}{2}\pi-\gamma < \phi \leqslant \pi), \end{aligned}\right\} \tag{6.9}$$

which, with $\hat{\chi}_i$ defined by (5.19), verifies (5.20*a*, *b*) for the semicircular obstacle. It is helpful to note that the two intervals of ϕ in (6.9) are characterized by opposite signs of $\cos(\phi+\gamma)$. With the aid of (5.18), the estimation of the series (6.2) is now complete. With the angle of incidence allowed to vary through acute angles, it follows that, if $0 \leqslant \tau \leqslant \frac{1}{2}\pi$,

$$\begin{aligned} \mu_0(a\phi;\tau) \sim &-\nu\cos(\phi+\tau)\,\mathrm{e}^{-\mathrm{i}\nu a\cos\phi+\tau)}\,(\tfrac{1}{2}\pi-\tau < \phi \leqslant \pi) \\ &+\nu\cos(\phi-\tau)\,\mathrm{e}^{-\mathrm{i}\nu a\cos\phi-\tau)}\,(\tfrac{1}{2}\pi+\tau < \phi \leqslant \pi), \end{aligned} \tag{6.10a}$$

where the intervals indicate whether the corresponding term is to be included in the asymptotic form of $\mu_0(a\phi;\tau)$. As τ increases towards $\frac{1}{2}\pi$, the first of these lit segments expands to the whole of Γ while the second shrinks to zero. The corresponding result for $\frac{1}{2}\pi \leqslant \tau \leqslant \pi$ is

$$\begin{aligned} \mu_0(a\phi;\tau) \sim &-\nu\cos(\phi+\tau)\,\mathrm{e}^{-\mathrm{i}\nu a\cos(\phi+\tau)} \quad (0 \leqslant \phi < \tfrac{3}{2}\pi-\tau) \\ &+\nu\cos(\phi-\tau)\,\mathrm{e}^{-\mathrm{i}\nu a\cos(\phi-\tau)} \quad (0 \leqslant \phi < \tau-\tfrac{1}{2}\pi). \end{aligned} \tag{6.10b}$$

The scattered field $\hat{\chi}$ in (6.3) can also be asymptotically estimated by the above methods but for $Y > a$, it is simpler to substitute (3.5) into the left-hand side of (6.3), yielding

$$\hat{\chi}(X, Y;\gamma) = -\frac{a}{2\pi}\int_C\int_0^{\pi}\mu_0(a\phi;\gamma)\,\mathrm{e}^{-\mathrm{i}\nu[(X-a\cos\phi)\cos\tau+Y\sin\tau]}\sin(\nu a\sin\phi\sin\tau)\,\mathrm{d}\tau\,\mathrm{d}\phi,$$

and then to use (5.18) and (6.9). The principal contribution to the resulting double integral arises from the two-dimensional saddle points at which the phase is stationary with respect to τ and ϕ. In complete agreement with geometrical optics, the values of ϕ and τ at the saddle point(s) define the points of contact with Γ of incident rays from one or both component plane waves of χ_i, and the angle(s) of reflexion towards (X, Y).

(*b*) *Determination of* μ *by using iterations of* (5.15)

When the finite shelf width is taken into account, the scattered field $\hat{\chi}$ induces an additional field

$$\int_0^{\pi}\mu_0(a\phi;\gamma)\,G_\chi^*(X-a\cos\phi, Y; a\sin\phi)\,a\,\mathrm{d}\phi,$$

which on substitution of (5.12) takes the form

$$\frac{\mathrm{i}a}{\pi}\int_C\int_0^{\pi}\mu_0(a\phi;\gamma)\,P(\tau)\,\chi_i(X, Y;\tau)\,\mathrm{e}^{\mathrm{i}\nu a\cos\phi\cos\tau}\sin(\nu a\sin\phi\sin\tau)\,\mathrm{d}\tau\,\mathrm{d}\phi$$

Then, with $P(\tau)$ expanded as in (3.21) and (2.29) substituted for χ_i, this additional field consists of terms of the form

$$-\frac{a}{2\pi}\int_C\int_0^{\pi}\mu_0(a\phi;\gamma)\,[p(\tau)]^n\,\mathrm{e}^{-\mathrm{i}\nu[(X-a\cos\phi)\cos\tau+(2nl\mp Y)\sin\tau]}\sin(\nu a\sin\phi\sin\tau)\,\mathrm{d}\tau\,\mathrm{d}\phi \quad (n \geqslant 1),$$

so the successive reflexions at $Y = 0$, l may be considered as fields scattered by circles of radius a, with centres at $(0, \pm 2nl)$, and modified by the reflexion coefficient $p(\tau)$. Identification of lit and dark segments shows that the first correction $\mu_1(a\phi; \gamma)$ to the source density is determined by reflexions travelling towards the shoreline, i.e. the circles in $Y > 0$. The spreading effect of the geometrical optics scattering by a circle means that $\mu_1(a\phi; \gamma)$ is smaller than $\mu_0(a\phi; \gamma)$ by a factor of order $[a/2(l-a)]^{\frac{1}{2}}$. To verify this, consider the expression obtained by substituting (5.14) in (5.16), which for the semicircle is

$$\mu_1(a\phi; \gamma) = \frac{\mathrm{i}a}{\pi}\int_C P(\tau)\,\mu_0(a\phi;\tau)\int_0^\pi \mu_0(a\phi';\gamma)\,\mathrm{e}^{\mathrm{i}\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,\mathrm{d}\phi'\,\mathrm{d}\tau. \quad (6.11)$$

The ϕ'-integral can be evaluated, to leading order, by substituting (6.10a) and seeking points of stationary phase in the various intervals of ϕ'. The result suggests a more direct method is available. First, substitution of the Fourier series (6.1) and (6.2) yields

$$\begin{aligned} a\int_0^\pi \mu_0(a\phi';\gamma)\,\mathrm{e}^{\mathrm{i}\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,\mathrm{d}\phi' \\ &= a\int_0^\pi \mu_0(a\phi';\gamma)\,\overline{\chi}_i(a\cos\phi', a\sin\phi';\tau)\,\mathrm{d}\phi' \\ &= 4\mathrm{i}\sum_{n=1}^{\infty}\frac{J_n(\nu a)}{H_n^{(2)}(\nu a)}\sin n\gamma\sin n\tau \\ &= \mathrm{i}[\overline{\kappa}^*(\nu a, \tau-\gamma) - \overline{\kappa}^*(\nu a, \tau+\gamma)], \end{aligned} \quad (6.12)$$

where

$$\kappa^*(\xi,\theta) = \sum_{n=0}^{\infty}\epsilon_n\frac{J_n(\xi)}{H_n^{(1)}(\xi)}\cos n\theta = \sum_{n=-\infty}^{\infty}\frac{J_n(\xi)}{H_n^{(1)}(\xi)}\mathrm{e}^{\mathrm{i}n\theta}.$$

Again with the methods of Nussenzveig's analysis, Poisson's sum formula gives

$$\begin{aligned}\kappa^*(\xi,\theta) &= \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_\lambda(\xi)}{H_\lambda^{(1)}(\xi)}\mathrm{e}^{\mathrm{i}\lambda(\theta+2m\pi)}\,\mathrm{d}\lambda \\ &= \sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{J_\lambda(\xi)}{H_\lambda^{(1)}(\xi)}\mathrm{e}^{\mathrm{i}\lambda(\theta+2m\pi)}\,\mathrm{d}\lambda + \sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{J_{-\lambda}(\xi)}{H_\lambda^{(1)}(\xi)}\mathrm{e}^{\mathrm{i}\lambda[-\theta+(2m+1)\pi]}\,\mathrm{d}\lambda.\end{aligned}$$

The identity

$$J_\lambda(\xi)\,\mathrm{e}^{\mathrm{i}\pi\lambda} - J_{-\lambda}(\xi) = \mathrm{i}\sin\pi\lambda\,H_\lambda^{(2)}(\xi)$$

assists in obtaining the simpler form

$$\kappa^*(\xi,\theta) = 2\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{J_\lambda(\xi)}{H_\lambda^{(1)}(\xi)}\mathrm{e}^{(2m+1)\mathrm{i}\pi\lambda}\cos\lambda(\pi-\theta)\,\mathrm{d}\lambda + \frac{1}{2}\int_{-\infty+\mathrm{i}\epsilon}^{\infty+\mathrm{i}\epsilon}\frac{H_\lambda^{(2)}(\xi)}{H_\lambda^{(1)}(\xi)}\mathrm{e}^{-\mathrm{i}\lambda\theta}\,\mathrm{d}\lambda.$$

By the arguments applied to $\kappa(\xi, \theta)$, the above series of integrals can, for $0 < \theta \leqslant \pi$, be expressed as a sum of exponentially decaying residues. The remaining integral is estimated by deforming the path of integration to pass through the saddle point $\lambda = \xi\cos\frac{1}{2}\theta$ located by using (6.8) which implies that, in the stated region,

$$H_\lambda^{(2)}(\xi)/H_\lambda^{(1)}(\xi) \sim \mathrm{i}\exp\{-2\mathrm{i}[(\xi^2-\lambda^2)^{\frac{1}{2}} - \lambda\arccos(\lambda/\xi)]\}.$$

Hence the function κ^*, which is even in θ, is given by

$$\kappa^*(\xi,\theta) \sim \tfrac{1}{2}[\pi\xi|\sin\tfrac{1}{2}\theta|]^{\frac{1}{2}}\exp[-2\mathrm{i}\xi|\sin\tfrac{1}{2}\theta| + \tfrac{1}{4}\pi\mathrm{i}]$$

except when $\sin \frac{1}{2}\theta$ is small. Finally, substitution in (6.12) yields, for real τ,

$$a\int_0^{\pi} \mu_0(a\phi';\gamma)\,\mathrm{e}^{\mathrm{i}\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,\mathrm{d}\phi' = 4\mathrm{i}\sum_{n=1}^{\infty}\frac{J_n(\nu a)}{H_n^{(2)}(\nu a)}\sin n\gamma\sin n\tau$$
$$\sim \tfrac{1}{2}(\nu a\pi)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\{|\sin\tfrac{1}{2}(\tau-\gamma)|^{\frac{1}{2}}\mathrm{e}^{2\mathrm{i}\nu a\sin\frac{1}{2}|\tau-\gamma|}-[\sin\tfrac{1}{2}(\tau+\gamma)]^{\frac{1}{2}}\mathrm{e}^{2\mathrm{i}\nu a\sin\frac{1}{2}(\tau+\gamma)}\}. \quad (6.13)$$

Since formulae (6.10a, b) and (6.13) are valid for real τ only, their substitution into (6.11) is necessarily restricted. However, without detailed consideration, the analytic continuation of these results into the complex τ-plane must be such as to allow $\mu_1(a\phi;\gamma)$, given by (6.11), to be estimated by the methods used in §3 to obtain the near field behaviour of G_χ^*. Then (6.10a, b) and (6.13) can be used to locate the saddle points, which are all in the interval $(0,\pi)$ of the real τ-axis, and to evaluate the corresponding contributions to $\mu_1(a\phi;\gamma)$. For each ϕ in $(0,\pi)$, a saddle point contribution of leading order can only occur at a value of τ within the interval specified by (6.10a, b). In this way it is verified, in agreement with geometrical optics, that $\hat{\mu}(a\phi;-\tau)$ does not contribute significantly, for any ϕ, while $\hat{\mu}_0(a\phi;\tau)$ does so except when ϕ is close to 0 or π.

Thus, on deforming the contour C to S in (6.11), with asymptotically small error, and expanding $P_+(\tau)$ as in (3.21), the terms obtained have phases

$$-\nu[2nl\sin\tau+a\cos(\phi+\tau)-2a\sin\tfrac{1}{2}(\tau\mp\gamma)]\quad(n\geqslant 1)$$

(provided $\frac{1}{2}\pi-\phi<\tau<\frac{3}{2}\pi-\phi$) which are stationary when

$$2nl\cos\tau-a\sin(\phi+\tau)-a\cos\tfrac{1}{2}(\tau\mp\gamma)=0.$$

With $a/2l$ regarded as a small parameter, this transcendental equation has approximate solution

$$\tau\sim\tfrac{1}{2}\pi-(a/2nl)[\cos\phi+\cos(\tfrac{1}{4}\pi\mp\tfrac{1}{2}\gamma)]\{1+(a/2nl)[\sin\phi+\tfrac{1}{2}\sin(\tfrac{1}{4}\pi\mp\tfrac{1}{2}\gamma)]\}.$$

For each $n\geqslant 1$ and choice of sign, corresponding to the two terms of (6.13), such a saddle point occurs provided

$$\phi>(a/2nl)[\cos\phi+\cos(\tfrac{1}{4}\pi\mp\tfrac{1}{2}\gamma)]>\phi-\pi, \quad (6.14)$$

where the interpretation of the inequalities must allow for the penumbra regions. Then, continuing to retain only the first term in $\tau-\frac{1}{2}\pi$, the dominant terms in the double integral (6.11) for μ_1 are, with respect to powers of $(\nu l)^{-1}$ and $(a/2l)$, given by

$$\mu_1(a\phi;\gamma)\sim\tfrac{1}{2}\nu\sin\phi\,\mathrm{e}^{\mathrm{i}\nu a\sin\phi}\sum_{n=1}^{\infty}[p_+(\tfrac{1}{2}\pi)]^n\mathrm{e}^{-2\mathrm{i}\nu nl}$$
$$\times\left[\left[\frac{\sin(\frac{1}{4}\pi+\frac{1}{2}\gamma)}{nl/a-\frac{1}{2}\sin\phi-\frac{1}{4}\sin(\frac{1}{4}\pi-\frac{1}{2}\gamma)}\right]^{\frac{1}{2}}\exp\left\langle\mathrm{i}\nu a\left\{2\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)-\frac{a}{4nl}[\cos\phi+\cos(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)]^2\right\}\right\rangle\right.$$
$$\left.-\left[\frac{\sin(\frac{1}{4}\pi-\frac{1}{2}\gamma)}{nl/a-\frac{1}{2}\sin\phi-\frac{1}{4}\sin(\frac{1}{4}\pi+\frac{1}{2}\gamma)}\right]^{\frac{1}{2}}\exp\left\langle\mathrm{i}\nu a\left\{2\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)-\frac{a}{4nl}[\cos\phi+\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)]^2\right\}\right\rangle\right] \quad (6.15)$$

subject to the restriction (6.14). Thus the source distribution $\mu_1(a\phi;\gamma)$ is, in contrast to $\mu_0(a\phi;\gamma)$, almost symmetric about $\phi=\frac{1}{2}\pi$ and asymptotically non-zero over nearly all of Γ. The saddle points are all close to $\tau=\frac{1}{2}\pi$ because only rays travelling almost offshore can be reflected back to the scattering obstacle. The vanishing of the first term of (6.13) near $\tau=\gamma$, which can be regarded as a 'shadow spot' with respect to the plane wave $\chi_i(x, y; -\gamma)$ defined by (5.19), is immaterial to expression (6.15).

The next iterate $\mu_2(a\phi; \gamma)$ of the source density is given, from (5.14) and (5.15), by

$$\mu_2(a\phi;\gamma) = \frac{ia}{\pi}\int_C P(\tau)\,\mu_0(a\phi;\tau)\int_0^\pi \mu_1(a\phi';\gamma)\,e^{i\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,d\phi'\,d\tau,$$

which, by substitution of (6.11), can be rearranged into another form resembling (6.11), namely

$$\mu_2(a\phi;\gamma) = \frac{ia}{\pi}\int_C P(\tau)\,\mu_1(a\phi;\tau)\int_0^\pi \mu_0(a\phi';\gamma)\,e^{i\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,d\phi'\,d\tau \tag{6.16}$$

Since the saddle points are again close to $\tau = \frac{1}{2}\pi$, the 'shadow spot' effect means that the form of (6.15) appropriate for use in (6.16) is

$$\mu_1(a\phi;\tau) \sim \tfrac{1}{2}\nu\sin\phi\,e^{i\nu a[\sin\phi+2\sin(\frac{1}{4}\pi+\frac{1}{2}\tau)]}\sum_{n=1}^{\infty}[p_+(\tfrac{1}{2}\pi)]^n\left(\frac{a}{nl}\right)^{\frac{1}{2}}\exp[-2i\nu nl-(i\nu a^2/4nl)\cos^2\phi]$$
$$(\phi > (a/2l)\cos\phi > \phi-\pi)$$
$$= \nu\,e^{2i\nu a\sin(\frac{1}{4}\pi+\frac{1}{2}\tau)}Z(\phi), \tag{6.17}$$

where

$$Z(\phi) = \tfrac{1}{2}e^{i\nu a\sin\phi}\sin\phi\sum_{n=1}^{\infty}[p(\tfrac{1}{2}\pi)]^n\left(\frac{a}{nl}\right)^{\frac{1}{2}}e^{-2i\nu nl-(i\nu a^2/4nl)\cos^2\phi}. \tag{6.18}$$

The separated form (6.17) is certainly advantageous and, when substituted with (6.13) into (6.16), yields

$$\frac{\mu_2(a\phi;\gamma)}{Z(\phi)} \sim \frac{i\nu}{\pi}\int_S\sum_{m=1}^{\infty}[p(\tfrac{1}{2}\pi)]^m e^{-2i\nu ml\sin\tau}e^{2i\nu a\sin(\frac{1}{4}\pi+\frac{1}{2}\tau)},$$
$$\times\tfrac{1}{2}(\nu a\pi)^{\frac{1}{2}}e^{\frac{1}{4}\pi i}\{|\sin\tfrac{1}{2}(\tau-\gamma)|^{\frac{1}{2}}e^{2i\nu a\sin\frac{1}{2}|\tau-\gamma|}-[\sin\tfrac{1}{2}(\tau+\gamma)]^{\frac{1}{2}}e^{2i\nu a\sin\frac{1}{2}(\tau+\gamma)}\}\,d\tau$$

after the contour has been deformed to S and $P(\tau)$ expanded as previously. The phase functions in this integral are stationary when

$$\tau \sim \tfrac{1}{2}\pi-(a/2ml)\cos(\tfrac{1}{4}\pi\mp\tfrac{1}{2}\gamma),$$

and the corresponding saddle point contributions yield

$$\frac{\mu_2(a\phi;\gamma)}{Z(\phi)} \sim \tfrac{1}{2}\nu\,e^{2i\nu a}\sum_{m=1}^{\infty}[p(\tfrac{1}{2}\pi)]^m\left(\frac{a}{ml}\right)^{\frac{1}{2}}e^{-2i\nu ml}$$
$$\times\langle[\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)^{\frac{1}{2}}]\exp\{i\nu a[2\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)-(a/4ml)\cos^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)]\}$$
$$-[\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)]^{\frac{1}{2}}\exp\{i\nu a[2\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)-(a/4ml)\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)]\}\rangle$$
$$\sim e^{i\nu a}\mu_1(\tfrac{1}{2}\pi a;\gamma), \tag{6.19}$$

by comparison with (6.15). Thus $\mu_2(a\phi;\gamma)$ is, to leading order in $(\nu l)^{-1}$ and a/l, of separated form with the offshore angle $\tau = \frac{1}{2}\pi$ appearing as a preferred direction not only in $\mu_1(\frac{1}{2}\pi a;\gamma)$ but also in the construction of $Z(\phi)$ in (6.17). However, the form of the iterative sequence of source density functions does not become apparent until the next step has been completed. Now

$$\mu_3(a\phi;\gamma) = \frac{ia}{\pi}\int_C P(\tau)\,\mu_0(a\phi;\tau)\int_0^\pi \mu_2(a\phi';\gamma)\,e^{i\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,d\phi'\,d\tau,$$

and by rearrangement with the use of (6.11) it readily follows that the equation for $\mu_3(a\phi;\gamma)$ corresponding to (6.16) for $\mu_2(a\phi;\gamma)$ is

$$\mu_3(a\phi;\gamma) = \frac{ia}{\pi}\int_C P(\tau)\,\mu_2(a\phi;\tau)\int_0^\pi \mu_0(a\phi';\gamma)\,e^{i\nu a\cos\phi'\cos\tau}\sin(\nu a\sin\phi'\sin\tau)\,d\phi'\,d\tau$$

By virtue of (6.19), the arguments applied to (6.16) now imply that

$$\mu_3(a\phi;\gamma) \sim e^{2i\nu a}Z(\phi)\,Z(\tfrac{1}{2}\pi)\,\mu_1(\tfrac{1}{2}\pi a;\gamma). \tag{6.20}$$

Comparison with (6.19) indicates that the iterative sequence is such that

$$\begin{aligned}\mu_{n+1}(a\phi;\gamma) &\sim [e^{i\nu a}Z(\tfrac{1}{2}\pi)]^{n-1}\mu_2(a\phi;\gamma) \quad (n\geqslant 1)\\ &\sim e^{in\nu a}[Z(\tfrac{1}{2}\pi)]^{n-1}Z(\phi)\,\mu_1(\tfrac{1}{2}\pi a;\gamma) \quad (n\geqslant 1).\end{aligned}$$

Hence the total source density is asymptotically given by

$$\mu(a\phi;\gamma) = \sum_{n=0}^{\infty}\mu_n(a\phi;\gamma) \sim \mu_0(a\phi;\gamma)+\mu_1(a\phi;\gamma)+\frac{e^{i\nu a}Z(\phi)\,\mu_1(\tfrac{1}{2}\pi a;\gamma)}{1-Z(\tfrac{1}{2}\pi)\,e^{i\nu a}}, \tag{6.21}$$

where μ_0, μ_1 and Z are given by (6.10a), (6.15) and (6.18) respectively. In the last term, the denominator is

$$\begin{aligned}1-Z(\tfrac{1}{2}\pi)\,e^{i\nu a} &= 1-\tfrac{1}{2}e^{2i\nu a}\sum_{n=1}^{\infty}[p(\tfrac{1}{2}\pi)]^n\left(\frac{a}{nl}\right)^{\frac{1}{2}}e^{-2i\nu nl}\\ &= 1-\tfrac{1}{2}(a/l)^{\frac{1}{2}}p(\tfrac{1}{2}\pi)\,e^{-2i\nu(l-a)}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l},\tfrac{1}{2},1],\end{aligned} \tag{6.22}$$

where Φ is defined by (3.23). In these calculations, the degree of approximation with respect to the ratio a/l is that terms of order $(a/l)^2$ relative to $\mu_0(a\phi;\gamma)$ have been neglected, νa being regarded as equal to eight in the exponential functions. In this respect, there is no reason to proceed beyond the iterate $\mu_3(a\phi;\gamma)$, given by (6.20), and the result (6.21) is equivalent to terminating the series at $n = 3$. However, the retention of further terms illustrates how the higher iterates are due to the presence of rays that are almost trapped on the offshore line $x = 0$, $a\leqslant y\leqslant l$. In (6.22), $p(\frac{1}{2}\pi)$ is the reflexion coefficient, and $2\nu(l-a)$ the phase change per cycle of such a trapped ray. There is an obvious comparison with the argument $p(\frac{1}{2}\pi)\,e^{-2i\nu l}$ of the generalized ζ-function Φ which appeared in (3.24) in respect of rays propagating exactly offshore from a source.

The source density $\mu(a\phi;\gamma)$ having been estimated, the scattered field is given, from (5.5), by

$$\chi(x,y) = \int_0^\pi \mu(a\phi;\gamma)\,G_\chi(x-a\cos\phi,y;a\sin\phi)\,a\,d\phi, \tag{6.23}$$

which in the near field consists of a complicated pattern of reflected rays as indicated above. However, in the far field, substitution of (3.20a, b) in (6.23) yields

$$\left.\begin{aligned}\chi(x,y) &\sim 2i\sum_{m=1}^{N}\frac{\Pi(\gamma_m,\gamma)}{E(\gamma_m)}e^{-i\nu x\cos\gamma_m}\sin(\nu y\sin\gamma_m) \quad \text{as} \quad x\to\infty,\\ \chi(x,y) &\sim -2i\sum_{m=1}^{N}\frac{\Pi(\pi-\delta_m,\gamma)}{E(\pi-\delta_m)}e^{i\nu x\cos\delta_m}\sin(\nu y\sin\delta_m) \quad \text{as} \quad x\to-\infty,\end{aligned}\right\} \tag{6.24}$$

where

$$\Pi(\tau,\gamma) = a\int_0^\pi \mu(a\phi;\gamma)\,e^{i\nu a\cos\phi\cos\tau}\sin(\nu a\sin\phi\sin\tau)\,d\phi. \tag{6.25}$$

If a sequence $\{\Pi_n(\tau, \gamma); n \geqslant 0\}$ is related to $\{\mu_n(a\phi; \gamma); n \geqslant 0\}$ in the obvious manner, then $\Pi_0(\tau, \gamma)$ is given by (6.13), and (5.13) implies that $\Pi(\tau, \gamma)$ satisfies the integral equation

$$\Pi(\tau, \gamma) - \Pi_0(\tau, \gamma) = \frac{\mathrm{i}}{\pi}\int_C P(\tau')\,\Pi_0(\tau, \tau')\,\Pi(\tau', \gamma)\,\mathrm{d}\tau'.$$

However, an iterative solution of this equation is a more difficult calculation than the evaluation of the integrals obtained by direct substitution of (6.21) into (6.25). The values of E and the complex Π_0 (in radian polar form), displayed in table 2, are calculated from (3.19) and (6.13) for the modes listed in table 1 and suggest that the greater response of the higher modes in (6.24) is due to both the E and Π functions.

Table 2

m	$E(\gamma_m)$	$2\mathrm{i}\Pi_0(\gamma_m, \gamma)$	$-E(\Pi-\delta_m)$	$2\mathrm{i}\Pi_0(\pi-\delta_m, \gamma)$
1	45.32	(1.747, 1.158)	39.00	(0.762, −2.730)
2	38.98	(2.963, 2.750)	37.49	(1.526, −2.882)
3	35.09	(3.750, −2.179)	34.83	(2.296, 3.130)
4	30.70	(4.392, −0.732)	30.73	(3.076, 2.701)
5	24.70	(4.925, 0.915)	24.46	(3.882, 2.021)
6	13.38	(5.321, 3.140)	13.30	(4.765, 0.720)

7. The slender obstacle

Suppose that the curve Γ is given by

$$y = \epsilon L\hat{Y}(x/L) \quad (|x| \leqslant L, \hat{Y}(\pm 1) = 0), \tag{7.1}$$

where ϵ is a small parameter and $\hat{Y}$ has maximum value one and derivatives of order unity. Then the principles of geometrical optics suggest that the portion of the incident shelf wave that, in the absence of the obstacle, would have been reflected from the segment $|x| < L$ of the shoreline, will now be slightly 'fanned out', but will not be reflected back to the obstacle provided, to leading order, $L < l \cot \gamma$, which will be assumed to hold. Since the sharp corners of the obstacle region D lie on the shoreline where $\chi_i(x, y; \gamma)$ vanishes, no edge wave diffraction occurs and hence only the basic scattering problem need be solved since further iterates due to finite bl must be asymptotically zero.

The use of a source distribution on Γ, satisfying (5.7), is likely to involve the geometrical optics approach of Levy & Keller (1959). An alternative method of solving the basic scattering problem is to represent the scattered field as that due to singularities placed within D, as illustrated by (6.3). In the present case, the singularities may be confined to the x-axis, about which D is symmetric, and must, as in (6.3), be of odd order because condition (2.30) implies that all the fields due to even-order singularities are identically zero when the singularity is at the shoreline. Thus define, for $n \geqslant 0$,

$$\begin{aligned} G_\chi^{2n+1}(x, y) &= G_\chi^{0,\,2n+1}(x, y) + G_\chi^{*,\,2n+1}(x, y) \\ &= \left[\frac{\partial^{2n+1} G_\chi^0(x, y; Y)}{\partial Y^{2n+1}}\right]_{Y=0} + \left[\frac{\partial^{2n+1} G_\chi^*(x, y; Y)}{\partial Y^{2n+1}}\right]_{Y=0} \\ &= \left[\frac{\partial^{2n+1} G_\chi(x, y; Y)}{\partial Y^{2n+1}}\right]_{Y=0}. \end{aligned} \tag{7.2}$$

Then each function $G_\chi^{2n+1}(x, y)$ satisfies the same conditions as G_χ but with singularity at the origin, being given by $G_\chi^{0,\,2n+1}(x, y)$ which is a multiple of $H_{2n+1}^{(2)}(\nu\lambda)\sin(2n+1)\phi$. In particular, from (3.2),

$$G_\chi^{0,\,1}(x, y) = \tfrac{1}{2}\mathrm{i}\nu H_1^{(2)}(\nu\lambda)\sin\phi$$
$$\sim -\sin\phi/\pi\lambda = -y/\pi(x^2+y^2) \quad \text{as} \quad \lambda\to 0,$$

which corresponds to a dipole singularity.

A solution of the basic scattering problem, obtained by exploiting the slenderness parameter ϵ in (7.1), can now be written as

$$\sum_{n=0}^{\infty}\epsilon^{2n+1}\int_{-L}^{L}(L^2-X^2)^{2n+1}g_n(X,\epsilon)\,G_\chi^{0,\,2n+1}(x-X, y)\,\mathrm{d}X,$$

where the functions $g_n(X, \epsilon)$ are to be determined from the integral equation obtained from condition (2.32), namely

$$\sum_{n=0}^{\infty}\epsilon^{2n+1}\int_{-L}^{L}(L^2-X^2)^{2n+1}g_n(X,\epsilon)\,G_\chi^{0,\,2n+1}[x-X, \epsilon L\hat{Y}(x/L)]\,\mathrm{d}X = -\chi_i(x, \epsilon L\hat{Y}(x/L);\gamma). \quad (7.3)$$

Here the factors ϵ^{2n+1} anticipate the orders of magnitude of the density functions, and the factors $(L^2-X^2)^{2n+1}$ are required to allow repeated integration by parts. The details of the expansion of (7.3) in terms of the small parameter ϵ, and the subsequent determination of $\{g_n(x,\epsilon);\, n\geqslant 0\}$, follow those of Geer & Keller (1968), and since further iterates are asymptotically zero, the required scattered field $\chi(x, y)$ is given by

$$\chi(x,y) \sim \sum_{n=0}^{\infty}\epsilon^{2n+1}\int_{-L}^{L}(L^2-X^2)^{2n+1}g_n(X,\epsilon)\,G_\chi^{2n+1}(x-X, y)\,\mathrm{d}X. \quad (7.4)$$

With the aid of (7.2), substitution of (3.20 a, b) shows that the far field behaviour is such that

$$\chi(x,y) \sim 2\mathrm{i}\sum_{m=1}^{N}\sum_{n=0}^{\infty}\frac{(\epsilon\nu\sin\gamma_m)^{2n+1}}{E(\gamma_m)}(-1)^n\int_{-L}^{L}(L^2-X^2)^{2n+1}g_n(X,\epsilon)\,\mathrm{d}X$$
$$\times\,\mathrm{e}^{-\mathrm{i}\nu x\cos\gamma_m}\sin(\nu y\sin\gamma_m) \quad \text{as} \quad x\to\infty$$

$$\chi(x,y) \sim 2\mathrm{i}\sum_{m=1}^{N}\sum_{n=0}^{\infty}\frac{(\epsilon\nu\sin\delta_m)^{2n+1}}{E(\pi-\delta_m)}(-1)^{n+1}\int_{-L}^{L}(L^2-X^2)^{2n+1}g_n(X,\epsilon)\,\mathrm{d}X$$
$$\times\,\mathrm{e}^{\mathrm{i}\nu x\cos\delta_m}\sin(\nu y\sin\delta_m) \quad \text{as} \quad x\to-\infty.$$

The validity of the solution of (7.3) by means of slender body theory requires, as stated by Geer (1978) for the corresponding three-dimensional problem, that $\nu L\epsilon$ must be small. The numerical values of §4 have $\nu\approx 0.4\ \mathrm{km}^{-1}$ in which case $L\epsilon$, the maximum projection offshore of the obstacle, must be less than one kilometre.

Since the order-ϵ terms in (7.3) yield

$$g_0(x,\epsilon) \sim \frac{L\hat{Y}(x/L)}{L^2-x^2}\left(\frac{\partial\chi_i}{\partial y}\right)_{y=0},$$

it follows that the leading term of (7.4), namely the convolution integral

$$\epsilon\int_{-L}^{L}(L^2-X^2)\,g_0(X,0)\,G_\chi^1(x-X,y)\,\mathrm{d}X$$

can be readily identified as the field calculated by Buchwald (1976) who expressed condition (2.32) as an approximate one on the shoreline and was then able to use a Fourier transform in the alongshore direction. His expression (17) is $(\partial\chi_i/\partial y)_{y=0}$ times the suppressed time factor times the inverse Fourier transform of the product of the transforms of $L\hat{Y}(x/L)$ and $G^1_\chi(x, y)$.

8. The offshore barrier

(a) Determination of an approximate μ_0

For the offshore barrier, the curve Γ is the repeated line $x = 0$, $0 \leqslant y \leqslant a$, and the scattered field is given by (5.1) with the source density $\mu(y;\gamma)$ determined by (5.4). The geometrical optics approximation is insufficient now for the estimation of the scattered field $\hat{\chi}(x, y;\gamma)$ in the $bl \to \infty$ limit because it cannot predict the edge-diffracted waves which are radiated from the sharp edge at $(0, a)$.

The construction of $\hat{\chi}$ must begin by taking due account of the singularity at this edge, near which the barrier appears to be semi-infinite, so that the dominant behaviour of $\hat{\chi}$ in this region may be expected to be like that of the well known solution for the scattering of a plane wave by a semi-infinite plane. This exact solution is given in various forms, all involving functions of period 4π, by Clemmow (1966), Jones (1964) and Morse & Feshbach (1953). The function

$$U(r\cos\varpi, r\sin\varpi;\tau) = -(1/\pi^{\frac{1}{2}})\,e^{-i(\nu r-\frac{1}{4}\pi)}\{F[(2\nu r)^{\frac{1}{2}}\sin\tfrac{1}{2}(\varpi+\tau)]+F[(2\nu r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\varpi-\tau)]\} \quad (8.1)$$

represents the scattered wave field $(-\frac{1}{2}\pi \leqslant \varpi \leqslant \frac{3}{2}\pi)$ produced when the plane wave $e^{-i\nu r\cos(\varpi+\tau)}$ travelling in a direction making an angle τ with the $\varpi = 0$ axis is incident on the semi-infinite 'soft' barrier $\varpi = -\frac{1}{2}\pi, \frac{3}{2}\pi, r > 0$. The function F in (8.1) is the complex Fresnel integral defined by

$$F(z) = e^{iz^2}\int_z^\infty e^{-i\xi^2}\,d\xi \quad (8.2)$$

(Clemmow 1966) and has the elementary properties

$$F(z)+F(-z) = \pi^{\frac{1}{2}}e^{i(z^2-\frac{1}{4}\pi)} \quad \text{for all } z, \quad (8.3)$$

$$F(z) \sim \frac{1}{2iz}\left(1-\frac{1}{2iz^2}+\dots\right) \quad \text{as} \quad |z|\to\infty\left(-\tfrac{3}{4}\pi < \arg z < \tfrac{1}{4}\pi\right). \quad (8.4)$$

Since only odd powers of z occur in (8.4), the relation (8.3) implies that as $|z|\to\infty$ in the other half-plane, the asymptotic form of $F(z)$ is the expansion (8.4) plus the exponential term which is dominant in the second and fourth quadrants and recessive in the first and third quadrants. The Stokes lines $\arg z = -\frac{3}{4}\pi, \frac{1}{4}\pi$ are the lines of steepest descent of e^{iz^2}.

The plane wave representation is preferred here, as in the construction of the Green function in §3 and the solution for the semicircular headland in §6, because it is the most amenable to the application of the conditions (2.31) at the shelf–ocean boundary $y = l$. The ease with which the shadow and lit regions may be identified is also attractive and, in the belief that the approximate solution presented here for scattering by a 'soft' strip is unavailable in the literature, the details are included for explanation and subsequent reuse in the succeeding approximations.

The function defined by (8.1) is evidently symmetric about $\tau = \frac{1}{2}\pi$, i.e.

$$U(r\cos\varpi, r\sin\varpi; \pi-\tau) = U(r\cos\varpi, r\sin\varpi, \tau), \tag{8.5}$$

since it is determined by the condition that it be equal to $-\mathrm{e}^{-\mathrm{i}\nu r\sin\tau}$ on $\varpi = -\frac{1}{2}\pi, \frac{3}{2}\pi$. Equations (8.3), (8.4) imply that, for large νr, the plane wave terms of (8.1) are confined to the regions where either the sine or cosine is negative, i.e. $-\frac{1}{2}\pi < \varpi < -\tau$ (shadow) and $\pi+\tau < \varpi < \frac{3}{2}\pi$ (reflexion) if the wave is incident from the left ($|\tau| < \frac{1}{2}\pi$), and $-\frac{1}{2}\pi < \varpi < \tau-\pi$ (reflexion) and $2\pi-\tau < \varpi < \frac{3}{2}\pi$ (shadow) if the wave is incident from the right ($\frac{1}{2}\pi < \tau < \frac{3}{2}\pi$), with the function F providing smooth transitions. The remaining terms, with exponential factor $\mathrm{e}^{-\mathrm{i}\nu r}$, correspond to edge diffraction, i.e. waves travelling in all radial directions away from the edge $r = 0$. For the wave approaching broadside on ($\tau = \frac{1}{2}\pi$),

$$U(r\cos\varpi, r\sin\varpi; \tfrac{1}{2}\pi) = -(2/\pi^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}(\nu r-\frac{1}{4}\pi)}F[(2\nu r)^{\frac{1}{2}}\sin(\tfrac{1}{2}\varpi+\tfrac{1}{4}\pi)] \tag{8.6}$$

which, according to (8.4), consists only of edge-diffracted waves except in transition regions either side of the semi-infinite plane. The function U is undefined at $\tau = -\frac{1}{2}\pi, \frac{3}{2}\pi$, a property that considerably complicates the subsequent argument.

The complex Fresnel integral is the canonical form of a plane wave representation with a simple pole. According to Clemmow (1966, equation (3.65)),

$$\int_{S(\theta)} \sec\tfrac{1}{2}(\tau-\tau_0)\,\mathrm{e}^{-\mathrm{i}\nu r\cos(\theta-\tau)}\,\mathrm{d}\tau = \mp 4\pi^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}(\nu r+\frac{1}{4}\pi)}F[\pm(2\nu r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\tau_0)], \tag{8.7}$$

where $S(\theta)$ is the contour shown in figure 2*b*, and the upper or lower sign is taken according as $\cos\frac{1}{2}(\theta-\tau_0)$ is positive or negative. Evidently the sign changes in (8.7) occur when θ coincides with a pole of the integrand, i.e. the transition region is characterized by the saddle point being close to the pole.

As a first step in constructing $\hat{\chi}$, the singular behaviour near the sharp edge at $(0, a)$ can be included by means of the function $V(x, y; \gamma)$, defined as the scattered field produced when condition (5.3) is applied on the semi-infinite barrier $x = 0$, $-\infty < y < a$. Then comparison of (2.29) and (8.1) shows that

$$V(x, y; \gamma) = (1/2\mathrm{i})\,[\mathrm{e}^{\mathrm{i}\nu a\sin\gamma}U(x, y-a; \gamma) - \mathrm{e}^{-\mathrm{i}\nu a\sin\gamma}U(x, y-a; -\gamma)]. \tag{8.8}$$

The non-vanishing of V at $y = 0$ can be remedied by subtracting $V(x, -y; \gamma)$, which is equivalent to adding the scattered field when the barrier is at $x = 0$, $-a < y < \infty$. Then (2.30) is satisfied but the field is equal to $-\chi_i$ at $x = 0$, $|y| < a$ and has discontinuous x-derivative at $x = 0$, $|y| > a$, arising principally from total reflexion. These difficulties are essentially overcome by subtracting the scattered field due to an infinite barrier at $x = 0$, namely $-\chi_i(|x|, y; \gamma)$. Hence the field

$$\chi^*(x, y; \gamma) = V(x, y; \gamma) - V(x, -y; \gamma) + \chi_i(|x|, y; \gamma) \tag{8.9}$$

satisfies (5.3) and, by elementary consideration of the shadow and reflexion regions described above, is seen to have a beamlike scattering pattern as shown in figure 3. Evidently χ^* is even in x, odd in y and its x-derivative at $x = 0$, $|y| > a$ must be small since edge diffracted waves propagate radially. Let the function $\Delta(y;\gamma)$ be defined by

$$\Delta(y;\gamma) = \left[\frac{\partial\chi^*}{\partial x}\right]_{x=0-}^{x=0+} = \frac{\partial\chi^*}{\partial x}(0+, y;\gamma) - \frac{\partial\chi^*}{\partial x}(0-, y;\gamma) \tag{8.10}$$

$$= 2\frac{\partial\chi^*}{\partial x}(0+, y;\gamma).$$

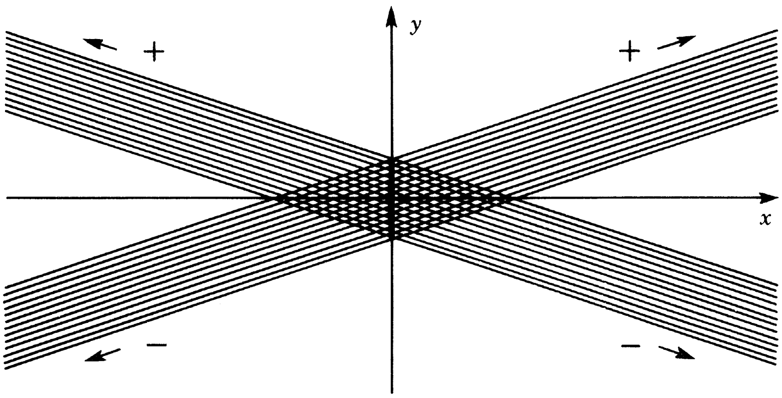

FIGURE 3. The beam-like scattering pattern of $\chi^*(x, y)$, which is even in x, odd in y. The signs indicate that for x positive, one or both of the component waves of χ_i in (2.29) may be annihilated, yielding shadow regions.

For $y > a$, $\partial V/\partial x$ is continuous and hence

$$\Delta(y;\gamma) = -2\frac{\partial V}{\partial x}(0+, -y;\gamma) + 2\frac{\partial \chi_i}{\partial x}(0, y;\gamma)$$

$$= -2\frac{\partial V}{\partial x}(0+, -y;\gamma) - 2\frac{\partial \chi_i}{\partial x}(0, -y;\gamma).$$

But from (8.1) and (8.3)

$$U(r\cos\varpi, r\sin\varpi;\tau) + e^{-i\nu r\cos(\varpi+\tau)} = (1/\pi^{\frac{1}{2}})\, e^{-i(\nu r-\frac{1}{4}\pi)}\{F[-(2\nu r)^{\frac{1}{2}}\sin\tfrac{1}{2}(\varpi+\tau)] - F[(2\nu r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\varpi-\tau)]\} \tag{8.11}$$

and thus

$$\left\{\frac{\partial}{\partial x}[U(x, y-a;\tau) + e^{-i\nu[x\cos\tau-(y-a)\sin\tau]}]\right\}_{\substack{x=0+\\ y<a}}$$

$$= \frac{1}{r}\left\{\frac{\partial}{\partial\varpi}[U(r\cos\varpi, r\sin\varpi;\tau) + e^{-i\nu r\cos(\varpi+\tau)}]\right\}_{\varpi=-\frac{1}{2}\pi}$$

$$= -(2\nu/\pi r)^{\frac{1}{2}} e^{-i(\nu r-\frac{1}{4}\pi)} F'[(2\nu r)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)]\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)$$

$$= -[\nu(1+\sin\tau)/\pi(a-y)]^{\frac{1}{2}} e^{-i[\nu(a-y)-\frac{1}{4}\pi]} F'[\nu^{\frac{1}{2}}(a-y)^{\frac{1}{2}}(1-\sin\tau)^{\frac{1}{2}}].$$

Hence, on substituting (2.29) and (8.8), it follows that, for $y > a$,

$$\Delta(y;\gamma) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \frac{e^{-i[\nu(a+y)+\frac{1}{4}\pi]}}{(a+y)^{\frac{1}{2}}}\{e^{i\nu a\sin\gamma}(1+\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a+y)^{\frac{1}{2}}(1-\sin\gamma)^{\frac{1}{2}}] - e^{-i\nu a\sin\gamma}(1-\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a+y)^{\frac{1}{2}}(1+\sin\gamma)^{\frac{1}{2}}]\} \tag{8.12}$$

Since $1-\sin\gamma$ is not small, the asymptotic formula (8.4) yields

$$\Delta(y;\gamma) \sim \frac{e^{-i[\nu(a+y)-\frac{1}{4}\pi]}}{2(\nu\pi)^{\frac{1}{2}}(a+y)^{\frac{3}{2}}}\left[\frac{e^{i\nu a\sin\gamma}(1+\sin\gamma)^{\frac{1}{2}}}{1-\sin\gamma} - \frac{e^{-i\nu a\sin\gamma}(1-\sin\gamma)^{\frac{1}{2}}}{1+\sin\gamma}\right] \quad (y > a),$$

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which expression is oscillatory and uniformly small, as expected, because it is due entirely to the edge-diffracted waves that travel along the barrier from the 'image' edge $(0, -a)$, or equivalently those that propagate along the barrier from the actual edge $(0, a)$ and are reflected back at the shoreline.

On the barrier, all three terms on the righthand side of (8.9) have discontinuous x-derivative. However, on writing

$$\chi^*(x, y; \gamma) = [V(x, y; \gamma) - \chi_i(|x|, -y; \gamma)] - [V(x, -y; \gamma) - \chi_i(|x|, y; \gamma)] - \chi_i(|x|, y; \gamma),$$

it readily follows from the above calculation that, for $0 < y < a$,

$$\begin{aligned}\Delta(y;\gamma) = {} & 2\mathrm{i}\nu\cos\gamma\sin(\nu y\sin\gamma) \\ & + \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}[\nu(a+y)+\frac{1}{4}\pi]}}{(a+y)^{\frac{1}{2}}} \{\mathrm{e}^{\mathrm{i}\nu a\sin\gamma}(1+\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a+y)^{\frac{1}{2}}(1-\sin\gamma)^{\frac{1}{2}}] \\ & \qquad - \mathrm{e}^{-\mathrm{i}\nu a\sin\gamma}(1-\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a+y)^{\frac{1}{2}}(1+\sin\gamma)^{\frac{1}{2}}]\} \\ & - \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}[\nu(a-y)+\frac{1}{4}\pi]}}{(a-y)^{\frac{1}{2}}} \{\mathrm{e}^{\mathrm{i}\nu a\sin\gamma}(1+\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a-y)^{\frac{1}{2}}(1-\sin\gamma)^{\frac{1}{2}}] \\ & \qquad - \mathrm{e}^{-\mathrm{i}\nu a\sin\gamma}(1-\sin\gamma)^{\frac{1}{2}} F'[\nu^{\frac{1}{2}}(a-y)^{\frac{1}{2}}(1+\sin\gamma)^{\frac{1}{2}}]\}. \end{aligned} \tag{8.13}$$

The first term is that predicted by geometrical optics, the second is small and the third is singular at $y = a$ because, from (8.2), $F'(0) = -1$.

To make use of the function χ^*, defined by (8.9), it is essential to remove the discontinuity in $\partial\chi^*/\partial x$ at $x = 0$, $y > a$, even at the expense of violating the barrier condition (5.3). The function $\hat{\chi}_0(x, y; \gamma)$ defined by

$$\hat{\chi}_0(x, y; \gamma) = \chi^*(x, y; \gamma) - \int_a^\infty \Delta(Y; \gamma)\, G_\chi^\circ(x, y; Y)\,\mathrm{d}Y, \tag{8.14}$$

where $G_\chi^\circ(x, y; Y)$, given by (5.8), is equal to the first two terms of (3.2), is clearly the appropriate modification of χ^* since, if $y > a$,

$$[\partial\hat{\chi}_0/\partial x]_{0-}^{0+} = [\partial\chi^*/\partial x]_{0-}^{0+} - \Delta(y; \gamma) = 0,$$

by virtue of (8.10) and the singular term $(2\pi)^{-1}\ln R$ in $\frac{1}{4}\mathrm{i}H_0^{(2)}(\nu R)$. Moreover, if, for X, Y fixed with $Y > 0$, Green's theorem is applied to $\chi^*(x, y; \gamma)$ and $\frac{1}{4}\mathrm{i}\{H_0^{(2)}(\nu R) - H_0^{(2)}(\nu R')\}$ in the first and second quadrants of the (x, y) plane successively, then it follows on subtraction that

$$\chi^*(x, y; \gamma) = \int_0^\infty \left[\frac{\partial\chi^*}{\partial x}(0+, Y; \gamma) - \frac{\partial\chi^*}{\partial x}(0-, Y; \gamma)\right] G_\chi^\circ(x, y; Y)\,\mathrm{d}Y.$$

Hence an alternative form of (8.14) is

$$\hat{\chi}_0(x, y; \gamma) = \int_0^a \Delta(Y; \gamma)\, G_\chi^\circ(x, y; Y)\,\mathrm{d}Y, \tag{8.15}$$

which indicates that the approximation $\hat{\chi}_0(x, y; \gamma)$ to the solution $\hat{\chi}(x, y; \gamma)$ of the basic scattering problem is generated by a source distribution on the barrier of density $\Delta(y; \gamma)$, which is therefore the corresponding approximation to $\mu_0(y; \gamma)$.

By using the representation (3.5), equations (8.14), (8.15) can be rewritten:

$$\hat{\chi}_0(x,y;\gamma) = \chi^*(x,y;\gamma)+\frac{1}{2\pi}\int_C \mathrm{e}^{-\mathrm{i}\nu x\cos\tau}\sin(\nu y\sin\tau)\int_a^\infty \varDelta(Y;\gamma)\,\mathrm{e}^{-\mathrm{i}\nu Y\sin\tau}\,\mathrm{d}Y\,\mathrm{d}\tau \quad (0\leqslant y\leqslant a), \tag{8.16}$$

$$\hat{\chi}_0(x,y;\gamma) = -\frac{1}{2\pi}\int_C \mathrm{e}^{-\mathrm{i}\nu(x\cos\tau+y\sin\tau)}\int_0^a \varDelta(Y;\gamma)\sin(\nu Y\sin\tau)\,\mathrm{d}Y\,\mathrm{d}\tau \quad (y\geqslant a). \tag{8.17}$$

In both integrals, an equivalent contour is S, and the symmetry of each contour about the point $\tau = \frac{1}{2}\pi$ allows x to be replaced by $|x|$, thus demonstrating the known evenness of $\hat{\chi}_0$ in x.

The Y-integrals can be evaluated straightforwardly. Substitution of (8.12) yields

$$\int_a^\infty \varDelta(Y;\gamma)\,\mathrm{e}^{-\mathrm{i}\nu Y\sin\tau}\,\mathrm{d}Y = \frac{2\nu}{\pi^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\cos\gamma[J(\nu-\nu\sin\gamma,\ \nu+\nu\sin\tau)-J(\nu+\nu\sin\gamma,\ \nu+\nu\sin\tau)], \tag{8.18}$$

where

$$J(\xi,\eta) = \frac{\mathrm{e}^{-\mathrm{i}\xi a}}{2\xi^{\frac{1}{2}}}\int_a^\infty \mathrm{e}^{-\mathrm{i}\eta Y}F'[\xi^{\frac{1}{2}}(a+Y)^{\frac{1}{2}}]\frac{\mathrm{d}Y}{(a+Y)^{\frac{1}{2}}} \quad (\mathrm{Im}\,\eta\leqslant 0).$$

Meanwhile, (8.13) implies that

$$\begin{aligned}\mathrm{i}\int_0^a \varDelta(Y;\gamma)\sin(\nu Y\sin\tau)\,\mathrm{d}Y = {}&\frac{\nu}{\pi^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\cos\gamma[I(\nu+\nu\sin\gamma,\ \nu+\nu\sin\tau)\\ &-I(\nu-\nu\sin\gamma,\ \nu+\nu\sin\tau)-I(\nu+\nu\sin\gamma,\ \nu-\nu\sin\tau)+I(\nu-\nu\sin\gamma,\ \nu-\nu\sin\tau)]\\ &+\cos\gamma\left\{\frac{\sin[\nu a(\sin\tau+\sin\gamma)]}{\sin\tau+\sin\gamma}-\frac{\sin[\nu a(\sin\tau-\sin\gamma)]}{\sin\tau-\sin\gamma}\right\},\end{aligned} \tag{8.19}$$

where

$$I(\xi,\eta) = \frac{\mathrm{e}^{-\mathrm{i}\xi a}}{2\xi^{\frac{1}{2}}}\int_{-a}^a \mathrm{e}^{-\mathrm{i}\eta Y}F'[\xi^{\frac{1}{2}}(a+Y)^{\frac{1}{2}}]\frac{\mathrm{d}Y}{(a+Y)^{\frac{1}{2}}}.$$

These integrals can be evaluated from the indefinite integral

$$\int \mathrm{e}^{-\mathrm{i}\eta y}F'(\xi^{\frac{1}{2}}y^{\frac{1}{2}})\frac{\mathrm{d}y}{y^{\frac{1}{2}}} = 2\mathrm{e}^{-\mathrm{i}\eta y}\left\{\frac{\xi^{\frac{1}{2}}F(\xi^{\frac{1}{2}}y^{\frac{1}{2}})-\eta^{\frac{1}{2}}F(\eta^{\frac{1}{2}}y^{\frac{1}{2}})}{\xi-\eta}\right\}+\text{const.} \tag{8.20}$$

obtained by integration by parts and use of (8.2). This shows that

$$\begin{aligned}J(\xi,\eta) &= -\frac{\mathrm{e}^{-\mathrm{i}(\xi+\eta)a}}{\xi-\eta}\left\{F[(2\xi a)^{\frac{1}{2}}]-\left(\frac{\eta}{\xi}\right)^{\frac{1}{2}}F[(2\eta a)^{\frac{1}{2}}]\right\}\\ &= -\frac{\mathrm{e}^{-\mathrm{i}(\xi+\eta)a}}{\xi-\eta}\left\{\frac{F'[(2\xi a)^{\frac{1}{2}}]-F'[(2\eta a)^{\frac{1}{2}}]}{2\mathrm{i}(2\xi a)^{\frac{1}{2}}}\right\},\end{aligned} \tag{8.21}$$

$$\begin{aligned}I(\xi,\eta) = {}&\frac{\mathrm{e}^{-\mathrm{i}(\xi+\eta)a}}{\xi-\eta}\left\{F[(2\xi a)^{\frac{1}{2}}]-\left(\frac{\eta}{\xi}\right)^{\frac{1}{2}}F[(2\eta a)^{\frac{1}{2}}]\right\}\\ &-\frac{\pi^{\frac{1}{2}}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2(\xi-\eta)}\left[1-\left(\frac{\eta}{\xi}\right)^{\frac{1}{2}}\right]\mathrm{e}^{-\mathrm{i}(\xi-\eta)a},\end{aligned} \tag{8.22}$$

since $F(0) = \frac{1}{2}\pi^{\frac{1}{2}}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}$, from (8.3).

The validity of (8.21) requires that

$$\lim_{y\to\infty}\mathrm{e}^{-\mathrm{i}\eta y}\left[F(\xi^{\frac{1}{2}}y^{\frac{1}{2}})-\left(\frac{\eta}{\xi}\right)^{\frac{1}{2}}F(\eta^{\frac{1}{2}}y^{\frac{1}{2}})\right] = 0,$$

which, according to (8.3) and (8.4), implies $-\frac{3}{4}\pi < \arg \xi^{\frac{1}{2}} < \frac{1}{4}\pi$ and $-\frac{1}{2}\pi \leqslant \arg \eta^{\frac{1}{2}} \leqslant 0$, the latter being consistent with the stated restriction on the definition of $J(\xi, \eta)$. These conditions are satisfied by taking the positive root $(2\nu)^{\frac{1}{2}} \sin(\frac{1}{4}\pi \pm \frac{1}{2}\gamma)$ of $\xi = \nu \pm \nu \sin\gamma$, as in (8.13), and choosing $\eta^{\frac{1}{2}} = (2\nu)^{\frac{1}{2}} \sin(\frac{1}{4}\pi + \frac{1}{2}\tau)$ when $\eta = \nu + \nu\sin\tau$ ($|\mathrm{Re}\,\tau| \leqslant \frac{1}{2}\pi$, $\mathrm{Im}\,\tau \leqslant 0$ or $\frac{1}{2}\pi \leqslant \mathrm{Re}\,\tau \leqslant \frac{3}{2}\pi$, $\mathrm{Im}\,\tau \geqslant 0$ as in unshaded regions of figure 2*b* with $\theta = \frac{1}{2}\pi$). The nearest branch cuts of $(1+\sin\tau)^{\frac{1}{2}}$, namely $S(-\frac{1}{2}\pi)$ and $S(\frac{3}{2}\pi)$, are obviously immaterial.

Since the range of integration is finite, there is no restriction on the validity of (8.22) which, due to (8.3), is independent of the choice of $\eta^{\frac{1}{2}}$. Again $\xi^{\frac{1}{2}}$ is positive, and $\eta^{\frac{1}{2}}$ is chosen as above when $\eta = \nu + \nu\sin\tau$ but some latitude is available when $\eta = \nu - \nu\sin\tau$. For the expression in square brackets involving Fresnel integrals to have no plane wave terms, as in (8.21), $(\nu - \nu\sin\tau)^{\frac{1}{2}}$ must have a branch cut along S with values $(2\nu)^{\frac{1}{2}} \sin(\frac{1}{4}\pi - \frac{1}{2}\tau)$ and $(2\nu)^{\frac{1}{2}} \sin(\frac{1}{2}\tau - \frac{1}{4}\pi)$ on the left and right of S respectively. If, on the other hand, it is desired to make evident the analytic dependence of I on η, then either choice of $(\nu - \nu\sin\tau)^{\frac{1}{2}}$ will suffice for the whole complex τ-plane with the symmetry about $\tau = \frac{1}{2}\pi$ and periodicity, period 2π, being recoverable from the functions of period 4π in τ by means of (8.3).

On substituting (8.21), (8.22) into (8.18), (8.19) respectively, it follows that

$$\int_a^\infty \Delta(Y;\gamma)\,\mathrm{e}^{-\mathrm{i}\nu Y\sin\tau}\,\mathrm{d}Y = 2\hat{f}(\tau,\gamma) - 2\hat{f}(\tau,-\gamma)$$
$$(|\mathrm{Re}\,\tau| \leqslant \tfrac{1}{2}\pi,\ \mathrm{Im}\,\tau \leqslant 0 \quad\text{or}\quad \tfrac{1}{2}\pi \leqslant \mathrm{Re}\,\tau \leqslant \tfrac{3}{2}\pi,\ \mathrm{Im}\,\tau \geqslant 0), \tag{8.23}$$

$$\int_0^a \Delta(Y;\gamma)\sin(\nu Y\sin\tau)\,\mathrm{d}Y = \hat{f}(\tau,\gamma) - \hat{f}(\tau,-\gamma) - \hat{f}(-\tau,\gamma) + \hat{f}(-\tau,-\gamma)$$
$$+\hat{g}(\tau,\gamma) - \hat{g}(\tau,-\gamma) - \hat{g}(-\tau,\gamma) + \hat{g}(-\tau,-\gamma), \tag{8.24}$$

where

$$\hat{f}(\tau,\gamma) = \frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\cos\gamma}{\pi^{\frac{1}{2}}(\sin\gamma - \sin\tau)}\left\{F[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi + \tfrac{1}{2}\gamma)] - \frac{\sin(\frac{1}{4}\pi + \frac{1}{2}\tau)}{\sin(\frac{1}{4}\pi + \frac{1}{2}\gamma)}F[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi + \tfrac{1}{2}\tau)]\right\}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\nu a(2+\sin\tau+\sin\gamma)}, \tag{8.25}$$

$$\hat{g}(\tau,\gamma) = \frac{\sin(\frac{1}{4}\pi - \frac{1}{2}\gamma)\sin(\frac{1}{4}\pi + \frac{1}{2}\tau)}{\mathrm{i}\,(\sin\gamma - \sin\tau)}\,\mathrm{e}^{\mathrm{i}\nu a(\sin\tau - \sin\gamma)}$$
$$= \tfrac{1}{4}\mathrm{i}[\operatorname{cosec}\tfrac{1}{2}(\tau-\gamma) + \sec\tfrac{1}{2}(\tau+\gamma)]\,\mathrm{e}^{\mathrm{i}\nu a(\sin\tau-\sin\gamma)}. \tag{8.26}$$

The definitions (8.25), (8.26) are valid throughout the complex τ-plane, but, as indicated above, the following rearrangement may be helpful:

$$\hat{f}(-\tau,\gamma) + \hat{g}(-\tau,\gamma) = \tfrac{1}{4}\mathrm{i}[\operatorname{cosec}\tfrac{1}{2}(\tau+\gamma) - \sec\tfrac{1}{2}(\tau-\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu a(\sin\tau+\sin\gamma)}$$
$$+\frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\cos\gamma}{\pi^{\frac{1}{2}}(\sin\gamma + \sin\tau)}\left\{F[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi + \tfrac{1}{2}\gamma)] - \frac{\sin(\frac{1}{2}\tau - \frac{1}{4}\pi)}{\sin(\frac{1}{4}\pi + \frac{1}{2}\gamma)}F[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)]\right\}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\nu a(2-\sin\tau+\sin\gamma)}. \tag{8.27}$$

Evidently the steepest descent path S is the Stokes line separating the two asymptotic forms.

The result (8.26) indicates that when (8.24) is substituted into (8.17), integrals of the type (8.7) are obtained. Although each side of (8.24) is analytic in τ, the combination of $\hat{g}$-functions consists of pairs of functions, e.g.

$$\tfrac{1}{4}\mathrm{i}\operatorname{cosec}\tfrac{1}{2}(\tau-\gamma)\,[\mathrm{e}^{\mathrm{i}\nu a(\sin\tau-\sin\gamma)} - \mathrm{e}^{-\mathrm{i}\nu a(\sin\tau-\sin\gamma)}],$$

that together are regular in τ but have different phases. This phenomenon corresponds to beam waves and it is of interest to verify that $\hat{\chi}_0(x, y;\gamma)$ has the same beam wave scattering pattern as $\chi^*(x, y; \gamma)$, illustrated in figure 3. Substitution of (8.23) in (8.16) shows immediately that the additional term consists only of small, edge-diffracted waves. It therefore remains to show that the beam wave structure of the right-hand side of (8.17) is identical to that of $\chi^*(x, y; \gamma)$, i.e. that if the $\hat{g}$-terms of (8.24) are substituted into (8.17), the τ-integral obtained is equal to $\chi^*(x, y; \gamma)$ when $y \geqslant a$.

Considering only $x \geqslant 0$ because χ^* and V are even in x, the condition $y \geqslant a$ implies, by comparison with (8.1), (8.8), that $0 \leqslant \varpi \leqslant \frac{1}{2}\pi$ for $V(x, y; \gamma)$ and $-\frac{1}{2}\pi \leqslant \varpi \leqslant 0$ for $V(x, -y; \gamma)$. Further, the angle τ in (8.1) must be set equal to $\pm\gamma$ in turn. For $U(r\cos\varpi, r\sin\varpi; \gamma)$ with $0 \leqslant \varpi \leqslant \frac{1}{2}\pi$, the arguments of the Fresnel integrals both remain positive and the canonical form (8.7) yields

$$U(r\cos\varpi, r\sin\varpi; \gamma) = \frac{\mathrm{i}}{4\pi}\int_{S(\varpi)} [\sec\tfrac{1}{2}(\tau-\pi+\gamma)+\sec\tfrac{1}{2}(\tau-\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\tau)}\,\mathrm{d}\tau$$
$$= \frac{\mathrm{i}}{4\pi}\int_{S} [\operatorname{cosec}\tfrac{1}{2}(\tau+\gamma)+\sec\tfrac{1}{2}(\tau-\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\tau)}\,\mathrm{d}\tau$$

since the deformation crosses no poles. The corresponding result for $U(r\cos\varpi, r\sin\varpi; -\gamma)$, with $0 \leqslant \varpi \leqslant \frac{1}{2}\pi$ still, is less easily derived because the shadow boundary $\varpi = \gamma$ means that the argument of the first Fresnel integral is negative for $0 \leqslant \varpi < \gamma$. Then, with the use of (8.3) and (8.7),

$$\frac{1}{\pi^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}(\nu r-\frac{1}{4}\pi)}F[(2\nu r)^{\frac{1}{2}}\sin\tfrac{1}{2}(\varpi-\gamma)] = \mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\gamma)} - \frac{1}{\pi^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}(\nu r-\frac{1}{4}\pi)}F[(2\nu r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\varpi+\pi-\gamma)]$$
$$= \mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\gamma)} - \frac{\mathrm{i}}{4\pi}\int_{S(\omega)}\operatorname{cosec}\tfrac{1}{2}(\tau-\gamma)\,\mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\tau)}\,\mathrm{d}\tau$$
$$= -\frac{\mathrm{i}}{4\pi}\int_{S}\operatorname{cosec}\tfrac{1}{2}(\tau-\gamma)\,\mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi-\tau)}\,\mathrm{d}\tau$$

since the deformation crosses the pole at $\tau = \gamma$. Thus the same result as for $\gamma < \varpi \leqslant \frac{1}{2}\pi$ is obtained and, on substitution of these plane wave representations of $U(r\cos\varpi, r\sin\varpi; \pm\gamma)$ in (8.8), it follows that

$$V(x, y; \gamma) = \frac{1}{2\pi\mathrm{i}}\int_{S}[\hat{g}(\tau, -\gamma)-\hat{g}(\tau, \gamma)]\,\mathrm{e}^{-\mathrm{i}\nu(x\cos\tau+y\sin\tau)}\,\mathrm{d}\tau \quad (x \geqslant 0, y \geqslant a) \qquad (8.28a)$$

where $\hat{g}$ is defined by (8.26). The corresponding results for $-\frac{1}{2}\pi \leqslant \varpi \leqslant 0$ are best obtained by observing that (8.3) enables (8.1) to be written as

$$U(r\cos\varpi, -r\sin\varpi; -\tau)+\mathrm{e}^{-\mathrm{i}\nu r\cos(\varpi+\tau)}$$
$$= -\frac{1}{\pi^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}(\nu r-\frac{1}{4}\pi)}\{-F[(2\nu r)^{\frac{1}{2}}\sin\tfrac{1}{2}(\varpi+\tau)]+F[(2\nu r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\varpi-\tau)]\}.$$

Hence, by comparison with the derivation of (8.28a), it follows that

$$V(x, -y; \gamma) = -\frac{1}{2\pi\mathrm{i}}\int_{S}[\hat{g}(-\tau, \gamma)-\hat{g}(-\tau, -\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu(x\cos\tau+y\sin\tau)}\,\mathrm{d}\tau+\chi_i(x, y; \gamma) \quad (x, y \geqslant 0), \qquad (8.28b)$$

and when (8.28a, b) are substituted into (8.9), the even function χ^* can be written in the form

$$\chi^*(x, y; \gamma) = -\frac{1}{2\pi\mathrm{i}}\int_{S}[\hat{g}(\tau, \gamma)-\hat{g}(\tau, -\gamma)-\hat{g}(-\tau, \gamma)+\hat{g}(-\tau, -\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu(x\cos\tau+y\sin\tau)}\,\mathrm{d}\tau \quad (y \geqslant a).$$

Thus, as anticipated, the expression (8.16) for $\hat{\chi}_0$ has the same beam wave scattering pattern as χ^*.

The order of magnitude of the difference $\hat{\chi}_0 - \chi^*$ on the barrier gives, since χ^* satisfies (5.3), a measure of the error involved in approximating $\hat{\chi}$ by $\hat{\chi}_0$. Substitution of (8.23) into (8.16) yields

$$\hat{\chi}_0(0, y; \gamma) = \chi^*(0, y; \gamma) + \frac{1}{\pi}\int_S [\hat{f}(\tau, \gamma) - \hat{f}(\tau, -\gamma)] \sin(\nu y \sin\tau)\,\mathrm{d}\tau \quad (0 \leqslant y \leqslant a). \quad (8.29)$$

Now, for all τ on S, the asymptotic expansion (8.4) can be used in (8.25) to obtain

$$\hat{f}(\tau, \gamma) \sim \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}} \sin(\frac{1}{4}\pi - \frac{1}{2}\gamma)\, \mathrm{e}^{-\mathrm{i}\nu a(2+\sin\tau+\sin\gamma)}}{8(\nu a)^{\frac{3}{2}}\pi^{\frac{1}{2}}(1+\sin\gamma)(1+\sin\tau)} \quad (8.30)$$

$$= \mathrm{e}^{-\mathrm{i}\nu a \sin\tau} \times \text{a slowly varying function of order } (\nu a)^{-\frac{3}{2}}.$$

The path S is the steepest descent path, yielding terms of order $(\nu a)^{-2}$ except for the component $\mathrm{e}^{\mathrm{i}\nu y \sin\tau}$ of $\sin(\nu y \sin\tau)$ when y is close to a, for which the corresponding integral becomes algebraic of order $(\nu a)^{-\frac{3}{2}}$ as $y \to a$. Thus (8.29) implies

$$\hat{\chi}_0(0, y; \gamma) = -\sin(\nu y \sin\gamma) + O\{(\nu a)^{-\frac{3}{2}}/[1 + \nu^{\frac{1}{2}}(a-y)^{\frac{1}{2}}]\} \quad (0 \leqslant y \leqslant a). \quad (8.31)$$

This result shows that $\hat{\chi}_0(x, y; \gamma)$ is a good approximation to the scattered field $\hat{\chi}(x, y; \gamma)$ in the $bl \to \infty$ limit, with, according to (8.15), $\Delta(y; \gamma)$ the corresponding approximation to $\mu_0(y; \gamma)$.

(b) *The scattered field due to* μ_0

On proceeding to consider the effects of the finite value of bl, the fields obtained by setting $\mu(y) = \Delta(y; \gamma)$ in (5.1) are given, after substitution of (3.10), (3.12), by

$$\int_0^a \Delta(Y; \gamma) G_\chi(x, y; Y)\,\mathrm{d}Y = \hat{\chi}_0(x, y; \gamma) + \frac{\mathrm{i}}{\pi}\int_S P(\tau)\,\mathrm{e}^{-\mathrm{i}\nu x\cos\tau} \sin(\nu y \sin\tau)$$
$$\times \int_0^a \Delta(Y; \gamma) \sin(\nu Y \sin\tau)\,\mathrm{d}Y\,\mathrm{d}\tau \quad (0 \leqslant y \leqslant l). \quad (8.32)$$

$$\int_0^a \Delta(Y; \gamma) G_\psi(x, y; Y)\,\mathrm{d}Y = -\frac{1}{2\pi}\int_S Q(\tau) \exp\{\mathrm{i}(b/\sigma - \nu\cos\tau)[x \pm \mathrm{i}(y-l)] + bl\}$$
$$\times \int_0^a \Delta(Y; \gamma) \sin(\nu Y \sin\tau)\,\mathrm{d}Y\,\mathrm{d}\tau \quad (y \geqslant l). \quad (8.33)$$

For $x \neq 0$, these fields can be expressed in terms of shelf wave modes by substituting (3.20a, b) in (8.32), or (3.25a, b) in (8.33). All the coefficients are determined by (8.24); for example, in the shelf region

$$\int_0^a \Delta(Y; \gamma) G_\chi(x, y; Y)\,\mathrm{d}Y \sim 2\mathrm{i}\sum_{m=1}^{6} \frac{\mathrm{e}^{-\mathrm{i}\nu x\cos\gamma_m} \sin(\nu y \sin\gamma_m)}{E(\gamma_m)} \int_0^a \Delta(Y; \gamma) \sin(\nu Y \sin\gamma_m)\,\mathrm{d}Y$$

as $x \to \infty$, with N set equal to 6 as in §4.

The scattering pattern of these fields can be investigated by using (8.24) again to evaluate the Y-integral, $P(\tau)$ being replaced in (8.32) by $P_+(\tau)$ as in (3.16) and $P_+(\tau)$ expanded on S

as in (3.21). Then, by deforming the path of integration of each term to its own steepest descent path, residue contributions are obtained from the integral in (8.32) as follows:

$$-(1/2\mathrm{i})\,[p(\gamma)]^n\,\mathrm{e}^{-\mathrm{i}\nu[x\cos\gamma+(2nl-y)\sin\gamma]} = -(1/2\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\nu(x\cos\gamma-y\sin\gamma)}$$
$$\text{if}\quad 2nl-y-a < x\tan\gamma < 2nl-y+a \quad (n\geqslant 1),$$
$$(1/2\mathrm{i})\,[p(\gamma)]^n\,\mathrm{e}^{-\mathrm{i}\nu[x\cos\gamma+(2nl+y)\sin\gamma]} = (1/2\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\nu(x\cos\gamma+y\sin\gamma)}$$
$$\text{if}\quad 2nl+y-a < x\tan\gamma < 2nl+y+a \quad (n\geqslant 1),$$
$$-(1/2\mathrm{i})\,[p(\pi-\gamma)]^n\,\mathrm{e}^{\mathrm{i}\nu[x\cos\gamma-(2nl-y)\sin\gamma]} = -(1/2\mathrm{i})\,\mathrm{e}^{\mathrm{i}\nu(x\cos\gamma+y\sin\gamma)+2\mathrm{i}n\epsilon}$$
$$\text{if}\quad 2nl-y-a < -x\tan\gamma < 2nl-y+a, \quad (n\geqslant 1)$$
$$(1/2\mathrm{i})\,[p(\pi-\gamma)]^n\,\mathrm{e}^{\mathrm{i}\nu[x\cos\gamma-(2nl+y)\sin\gamma]} = (1/2\mathrm{i})\,\mathrm{e}^{\mathrm{i}\nu(x\cos\gamma-y\sin\gamma)+2\mathrm{i}n\epsilon}$$
$$\text{if}\quad 2nl+y-a < -x\tan\gamma < 2nl+y+a \quad (n\geqslant 1).$$

The simplification for $x > 0$ is achieved because $\tau = \gamma$ is a pole of $P(\tau)$, defined by (3.11), and hence the phase change factors $p(\gamma)$ and $\mathrm{e}^{2\mathrm{i}\nu l\sin\gamma}$ $(=\mathrm{e}^{2\mathrm{i}\alpha l})$ cancel. For $x < 0$, the phase change factor is given, from (3.9), by

$$\mathrm{e}^{2\mathrm{i}\epsilon} = \frac{p(\pi-\gamma)}{p(\gamma)} = \frac{\mathrm{e}^{2c}-\mathrm{e}^{-2\mathrm{i}\gamma}}{\mathrm{e}^{2c}-\mathrm{e}^{2\mathrm{i}\gamma}} \quad (\epsilon = 37^\circ\ 27'),$$

and differs from unity because the wave totally reflected by the barrier cannot satisfy the conditions at $y = l$ and thus does not belong to a shelf wave mode.

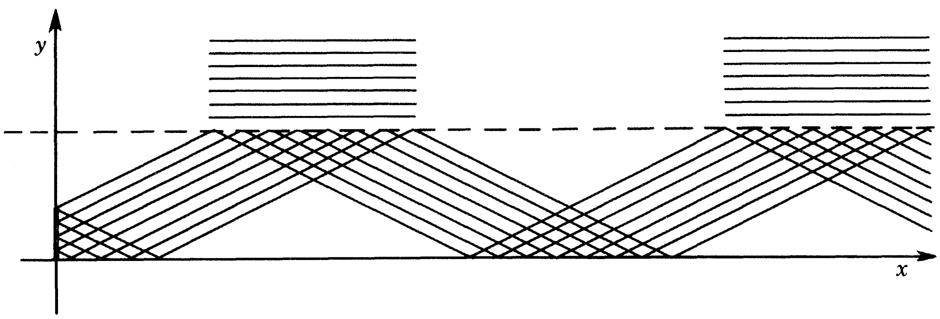

FIGURE 4. The beam-like scattering pattern, for $x > 0$, of the field generated in the shelf and ocean regions by the source density $\Delta(y, \gamma)$ and given by equations (8.32), (8.33).

The corresponding disturbances in the ocean region can most simply be found by combining the nth plane wave of the first or third sequence with the $(n-1)$th plane wave of the second or fourth sequence respectively and applying conditions (2.31). Then the contributions to the field (8.33) corresponding to the above residues in (8.32) are

$$-\exp\{(b/\sigma-\nu\cos\gamma)\,[\mathrm{i}x-(y-l)]+bl\}\sin\alpha l$$
$$\text{if}\quad (2n-1)\,l-a < x\tan\gamma < (2n-1)\,l+a \quad (n\geqslant 1),$$
$$-\exp\{(b/\sigma+\nu\cos\gamma)\,[\mathrm{i}x-(y-l)]+bl+(2n-1)\,\mathrm{i}\epsilon\}\sin(\alpha l+\epsilon)$$
$$\text{if}\quad (2n-1)\,l-a < -x\tan\gamma < (2n-1)\,l+a \quad (n\geqslant 1).$$

On comparison with (2.28), (2.29), these results show that, for $x > 0$, the beam wave of $\hat{\chi}_0$ is repeatedly reflected at $y = 0,\ l$ as indicated in figure 4 and in such a way as to annihilate the incident shelf wave mode in the ocean region $(y > l)$, together with one or both of the constituent incident waves in the shelf region ($0 < y < l$), forming regions of partial or total shadow. For $x < 0$, the results show that although there is a beam wave pattern which is the mirror image of figure 4, the reflected fields differ from those suggested by 'total reflexion', having phase changes in the shelf region and both phase and amplitude changes in the ocean. These beam wave patterns indicate regions where the field (8.32) has plane wave components, and the corresponding field (8.33) has an edge wave component, which are of the same order of magnitude as the incident shelf wave given by χ_i and ψ_i respectively. The edge-diffracted waves, which in the shelf region vary like $(\nu s)^{-\frac{1}{2}} e^{-i\nu s}$, where s denotes distance from $(0, a)$ along a ray, are propagated in all directions from the barrier edge and are then reflected at the shoreline $y = 0$ and reflected or refracted at $y = l$. Of particular significance is the ray directed along the line of the barrier, $x = 0$, since not only do all reflexions remain on this line but also it is evidently only this ray that necessitates the construction of the iterate $\mu_1(y; \gamma)$ of the source density, as described in §5. The violation of the boundary condition (5.3) due to the field

$$\int_0^a \Delta(Y; \gamma)\, G_\chi^*(x, y; Y)\, dY = \frac{i}{\pi} \int_C P(\tau)\, \chi_i(x, y; \tau) \int_0^a \Delta(Y; \gamma) \sin(\nu Y \sin\tau)\, dY\, d\tau, \quad (8.34)$$

which is additional to $\hat{\chi}_0(x, y; \gamma)$ in (8.32), is estimated by replacing $P(\tau)$ by $P_+(\tau)$ as in (3.16) and then using (3.22) or (3.24) to obtain

$$\int_0^a \Delta(Y; \gamma)\, G_\chi^*(0, y; Y)\, dY \sim \frac{i}{\pi} \int_S P_+(\tau) \sin(\nu y \sin\tau) \int_0^a \Delta(Y; \gamma) \sin(\nu Y \sin\tau)\, dY\, d\tau$$
$$\sim -\frac{p(\frac{1}{2}\pi)}{(\pi\nu l)^{\frac{1}{2}}} e^{-i(\frac{1}{4}\pi + 2\nu l)} \Phi[p(\tfrac{1}{2}\pi)\, e^{-2i\nu l}; \tfrac{1}{2}, 1] \sin\nu y \int_0^a \Delta(Y; \gamma) \sin(\nu Y)\, dY \quad (0 \leqslant y \leqslant a) \quad (8.35)$$

to leading order in a/l. The function Φ is defined by (3.23) and the Y-integral, given by (8.24), can be simplified by using (8.25), (8.26) and (8.30) to obtain

$$\int_0^a \Delta(Y; \gamma) \sin(\nu Y)\, dY = \tfrac{1}{2}i[\operatorname{cosec}(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma)\, e^{i\nu a(1-\sin\gamma)} - \operatorname{cosec}(\tfrac{1}{4}\pi + \tfrac{1}{2}\gamma)\, e^{i\nu a(1+\sin\gamma)}]$$
$$+ (e^{-\frac{1}{4}\pi i}/\pi^{\frac{1}{2}})\{F[2(\nu a)^{\frac{1}{2}} \sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma)]\, e^{-i\nu a(1-\sin\gamma)} \cot(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma)$$
$$- F[2(\nu a)^{\frac{1}{2}} \sin(\tfrac{1}{4}\pi + \tfrac{1}{2}\gamma)]\, e^{-i\nu a(1+\sin\gamma)} \cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\gamma)]\}$$
$$+ O[(\nu a)^{-\frac{3}{2}}] \quad (8.36)$$

The factor $(\nu l)^{-\frac{1}{2}}$ on the right-hand side of (8.35) shows that the approximate solution $\hat{\chi}_0(x, y; \gamma)$ is more helpful here than was the exact solution in the semicircular case, where the corresponding decay factor was $(a/l)^{\frac{1}{2}}$. However, such success is only available once because, in contrast to §6, the method, used to construct the approximations $\hat{\chi}_0$ and Δ to $\hat{\chi}$ and μ_0 respectively, cannot be used for subsequent iterates of the source density. The reason is that the function U, defined by (8.1), is undefined at $\tau = -\frac{1}{2}\pi$ and so the constructions V and thence χ^*, in (8.8) and (8.9) respectively, fail at the one angle that is evidently the most important. The alternative method used below is crude by comparison but, being devised so that the accuracy is maximized at the offshore direction, is sufficient for the present purposes.

The field (8.34) can, as in the description of (5.12), be regarded as the superposition of pairs of waves of the form $2i\chi_i(x, y; \tau)$, with the component $-e^{-i\nu(x\cos\tau + y\sin\tau)}$ being the reflexion at the shoreline of the incoming wave component $e^{-i\nu(x\cos\tau - y\sin\tau)}$. By pursuing this concept, the scattered field produced when this incoming wave is incident on the semi-infinite barrier

$x = 0, y \leqslant a$ is given, according to (8.1), by $e^{i\nu a \sin\tau} U(x, y-a; \tau)$, whose reflexion in the shoreline is $e^{i\nu a \sin\tau} U(x, -y-a; \tau)$. The failure of $U(x, -y-a; \tau)$ to account for the edge $(a, 0)$ should be immaterial for $\tau \neq \frac{1}{2}\pi$, because ray theory indicates that only the $\tau = \frac{1}{2}\pi$ ray can be incident on this edge. Indeed, if the field $e^{i\nu a \sin\tau}[U(x, y-a; \tau) - U(x, -y-a; \tau)]$ is inserted into the τ-integral in (8.34) instead of $2i\chi_i(x, y; \tau)$, then, for all $x \neq 0$, no plane waves are obtained, in agreement with the ideas of geometrical optics which indicate that only rays travelling along the line of the barrier are subject to further scattering. Thus the dominant value of τ is $\frac{1}{2}\pi$, for which U is given by (8.6).

(c) *Determination of further iterates*

Now consider the scattered field

$$U_1(x, y; \tau) = (1/2i)\, e^{i\nu a \sin\tau}\left\{U(x, y-a; \tau) - U(x, -y-a; \tau) + 2\int_a^\infty \left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} G^\circ_\chi(x, y; Y)\, dY\right\} \tag{8.37}$$

derived by modifying the above field to obtain one with continuous derivative at $x = 0$, $y > a$, as χ^* was adjusted to create $\hat{\chi}_0$ according to (8.11), (8.14). The expression corresponding to (8.15) is then

$$U_1(x, y; \tau) = -i e^{i\nu a \sin\tau}\int_0^a \left[\frac{\partial U(x, Y-a; \tau)}{\partial x} - \frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} G^\circ_\chi(x, y; Y)\, dY, \tag{8.38}$$

and a likely approximation to the next iterate of the scattered field, i.e. that due to the field (8.34), is therefore

$$\frac{i}{\pi}\int_C P(\tau)\int_0^a \Delta(Y'; \gamma)\sin(\nu Y' \sin\tau)\, dY'\, U_1(x, y; \tau)\, d\tau.$$

The corresponding approximation to the iterate $\mu_1(y; \gamma)$ is then given, on comparison of (8.15) and (8.38), by

$$\mu_1(y; \gamma) \sim \hat{\Delta}_1(y; \gamma) = \frac{1}{\pi}\int_C P(\tau)\, e^{i\nu a \sin\tau}\int_0^a \Delta(Y; \gamma)\sin(\nu Y \sin\tau)\, dY \times \left[\frac{\partial U(x, y-a; \tau)}{\partial x} - \frac{\partial U(x, -y-a; \tau)}{\partial x}\right]_{x=0+} d\tau \quad (0 < y < a). \tag{8.39}$$

The acceptance of these approximations depends on how accurately the above scattered field cancels the field (8.34) on the barrier.

The definition (8.1) implies that the derivatives of U in (8.37) and (8.39) are given by

$$\left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} = -\left[\frac{\nu}{2\pi(Y+a)}\right]^{\frac{1}{2}} e^{-i\nu(Y+a)+\frac{1}{4}\pi i}\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau) \times \{F'[(2\nu)^{\frac{1}{2}}(Y+a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)] + F'[(2\nu)^{\frac{1}{2}}(Y+a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)]\}\,(Y > -a). \tag{8.40}$$

On the barrier, (8.37) takes the form

$$U_1(0, y; \tau) + \chi_i(0, y; \tau) = -i e^{i\nu a \sin\tau}\int_a^\infty \left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} G^\circ_\chi(0, y; Y)\, dY$$
$$= \frac{i}{2\pi} e^{i\nu a \sin\tau}\int_C \sin(\nu y \sin\tau')\int_a^\infty \left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} e^{-i\nu Y \sin\tau'}\, dY\, d\tau' \quad (0 < y < a) \tag{8.41}$$

after (3.5) has been substituted. In the τ'-integral, both the path C and the integrand are symmetric about $\tau' = \frac{1}{2}\pi$, with the steepest descent path S being a Stokes line for the asymptotic expansions of the Fresnel integrals in (8.40). On the left side of S, (8.3) is used to write (8.40) as

$$\left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} = \mathrm{i}\nu \cos \tau\, \mathrm{e}^{-\mathrm{i}\nu(Y+a)\sin\tau}$$
$$-[2\nu/\pi(Y+a)]^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\nu(Y+a)+\frac{1}{4}\pi\mathrm{i}} \cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)F'[(2\nu)^{\frac{1}{2}}(Y+a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)],$$

which tends to $[2\nu/\pi(Y+a)]^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\nu(Y+a)+\frac{1}{4}\pi\mathrm{i}}$ as $\tau\to\frac{1}{2}\pi$, thus indicating that the failure to take due account of geometrical optics in the construction (8.37) is likely to be immaterial. Then

$$\int_a^\infty \left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} \mathrm{e}^{-\mathrm{i}\nu Y\sin\tau'}\,\mathrm{d}Y$$
$$= \mathrm{e}^{-\mathrm{i}\nu a\sin\tau}\cos\tau\left[\frac{\mathrm{e}^{-\mathrm{i}\nu a(\sin\tau+\sin\tau')}}{\sin\tau+\sin\tau'} - \frac{2\nu}{\pi^{\frac{1}{2}}}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}J(\nu-\nu\sin\tau, \nu+\nu\sin\tau')\right]$$
$$= \frac{\mathrm{e}^{-\mathrm{i}\nu a\sin\tau'}}{\sin\tau+\sin\tau'}\left\langle \mathrm{e}^{-2\mathrm{i}\nu a\sin\tau}\cos\tau - \mathrm{e}^{-\mathrm{i}(2\nu a+\frac{1}{4}\pi)}\frac{\cos(\frac{1}{4}\pi-\frac{1}{2}\tau)}{(\pi\nu a)^{\frac{1}{2}}}\{F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)]\right.$$
$$\left. -F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\tau')]\}\right\rangle,$$

since J is given by (8.21). This result is valid at least for τ, τ' each lying in one of the regions of validity of (8.23), i.e. when $\sin\tau$ and $\sin\tau'$ have non-positive imaginary parts. The expansion (8.4) is applicable to the last term and, by means of (8.3), the others can be written in a form that is obviously symmetric about $\tau = \frac{1}{2}\pi$, thus extending the result to the right side of S. Hence

$$\int_a^\infty \left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+} \mathrm{e}^{-\mathrm{i}\nu Y\sin\tau'}\,\mathrm{d}Y = -\frac{\mathrm{e}^{-\mathrm{i}[\nu a(2+\sin\tau')+\frac{1}{4}\pi]}\cos(\frac{1}{4}\pi-\frac{1}{2}\tau)}{2(\pi\nu a)^{\frac{1}{2}}(\sin\tau+\sin\tau')}$$
$$\times\{F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)]+F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau-\tfrac{1}{4}\pi)]+O(1/\nu a)\}. \quad (8.42)$$

Now, as τ varies from $-\mathrm{i}\infty$ to $\pi+\mathrm{i}\infty$ along S, the variable T defined by $\sin(\frac{1}{2}\tau-\frac{1}{4}\pi) = T\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ takes real values from $-\infty$ to ∞. Then

$$\int_S \mathrm{e}^{-2\mathrm{i}\nu l\sin\tau}\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau-\tfrac{1}{4}\pi)]\,\mathrm{d}\tau$$
$$= 2\mathrm{e}^{-\mathrm{i}(2\nu l-\frac{1}{4}\pi)}\int_{-\infty}^{\infty}\mathrm{e}^{-4\nu lT^2}F'[2(\nu a)^{\frac{1}{2}}T\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}]\,\mathrm{d}T$$
$$= -\left(\frac{\pi}{\nu l}\right)^{\frac{1}{2}}\frac{\mathrm{e}^{-\mathrm{i}(2\nu l-\frac{1}{4}\pi)}}{(1+a/l)}, \quad (8.43)$$

by integration by parts. Thus the factor $1+a/l$ measures the effect of the variations of the F'-function near the saddle point $\tau = \frac{1}{2}\pi$, i.e. approximating F' by $F'(0) = -1$ in the integral introduces an error of order a/l, which parameter does not appear in the phase function.

On substitution of (8.42) in (8.41), it now follows that

$$\frac{\mathrm{i}}{\pi}\int_C P(\tau)\int_0^a \Delta(Y';\gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y'[U_1(0,y;\tau)+\chi_i(0,y;\tau)]\,\mathrm{d}\tau$$
$$\sim\frac{\mathrm{e}^{-\mathrm{i}(2\nu a+\frac{1}{4}\pi)}}{2\pi^{\frac{5}{2}}(\nu a)^{\frac{1}{2}}}\int_S P_+(\tau)\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau)F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau-\tfrac{1}{4}\pi)]\int_S\frac{\mathrm{e}^{\mathrm{i}\nu a(\sin\tau-\sin\tau')}}{\sin\tau+\sin\tau'}\sin(\nu y\sin\tau')$$
$$\times\int_0^a \Delta(Y';\gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y'\,\mathrm{d}\tau'\,\mathrm{d}\tau$$

$$\sim -\frac{p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-\mathrm{i}\nu(2l+a)}}{2\pi^{\frac{3}{2}}\nu(al)^{\frac{1}{2}}}\int_S \frac{\mathrm{e}^{-\mathrm{i}\nu a\sin\tau'}\sin(\nu y\sin\tau')}{1+\sin\tau'}\int_0^a \Delta(Y';\gamma)\sin(\nu Y')\,\mathrm{d}Y'\,\mathrm{d}\tau'\,\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1] \qquad (0<y<a) \qquad (8.44)$$

to leading order in a/l. The Y-integral is given by (8.36), the τ' integral is essentially like that in (8.29) and the τ-integral has been estimated by the familiar methods of §3.

The result (8.44) measures the accuracy of the approximation

$$\frac{\mathrm{i}}{\pi}\int_C P(\tau)\int_0^a \Delta(Y';\gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y'\,U_1(x,y;\tau)\,\mathrm{d}\tau$$

to the scattered field produced when the field (8.34) is incident on the barrier. Comparison with (8.35) shows that the error is relatively of order $(\nu a)^{-\frac{1}{2}}$, which is much larger than that achieved for $\hat{\chi}_0$ in (8.31). However, since the dependence on y of the right-hand side of (8.41) is via $\chi_i(0,y;\tau')$ in a τ'-integration along C, comparison with (8.34) suggests that a more accurate approximation can be obtained by repeating the above process. Thus, by virtue of (8.37), define

$$U_2(x,y;\tau) = U_1(x,y;\tau)+\frac{\mathrm{i}}{2\pi}\mathrm{e}^{\mathrm{i}\nu a\sin\tau}\int_C U_1(x,y;\tau')\int_a^\infty\left[\frac{\partial U(x,-Y-a;\tau)}{\partial x}\right]_{x=0+} \times \mathrm{e}^{-\mathrm{i}\nu Y\sin\tau'}\,\mathrm{d}Y\,\mathrm{d}\tau'. \qquad (8.45)$$

Repeated substitution of (8.41) then shows that, on the barrier,

$$U_2(0,y;\tau)+\chi_i(0,y;\tau) = -\frac{1}{4\pi^2}\mathrm{e}^{\mathrm{i}\nu a\sin\tau}\int_C\int_C \sin(\nu y\sin\tau'') \times\int_a^\infty\left[\frac{\partial U(x,-Y'-a;\tau')}{\partial x}\right]_{x=0+}\mathrm{e}^{-\mathrm{i}\nu Y'\sin\tau''}\,\mathrm{d}Y'\int_a^\infty\left[\frac{\partial U(x,-Y-a;\tau)}{\partial x}\right]_{x=0+} \times \mathrm{e}^{-\mathrm{i}\nu(Y-a)\sin\tau'}\,\mathrm{d}Y\,\mathrm{d}\tau'\,\mathrm{d}\tau''.$$

The Y and Y' integrals are given by (8.42), and the resulting τ'-integral is

$$\int_S\left\{F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau')]+F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)]+O\left(\frac{1}{\nu a}\right)\right\}\frac{\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau')\,\mathrm{d}\tau'}{(\sin\tau+\sin\tau')(\sin\tau'+\sin\tau'')}$$

$$= \int_S\left\{2F'[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau')]+\frac{1}{4\mathrm{i}\nu a\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau')}\right\}\frac{\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\tau')\,\mathrm{d}\tau'}{(\sin\tau+\sin\tau')(\sin\tau'+\sin\tau'')}$$

$$= \int_{-\infty}^{\infty}\left\{F'[2(\nu a)^{\frac{1}{2}}T\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}]+\frac{1}{8\nu a(\mathrm{i}+T^2)}\right\}\frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,\mathrm{d}T}{[\sin^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\tau)-\mathrm{i}T^2][\sin^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\tau'')-\mathrm{i}T^2]}$$

$$= O(1/\nu a)$$

after the term involving the Fresnel integral has been integrated by parts. Consequently $[U_2(0,y;\tau)+\chi_i(0,y;\tau)]$ is of the same form, on the barrier, as $[U_1(0,y;\tau)+\chi_i(0,y;\tau)]$ but smaller by a factor of order $(\nu a)^{-\frac{3}{2}}$. Comparison with (8.44) then yields

$$\frac{\mathrm{i}}{\pi}\int_C P(\tau)\int_0^a \Delta(Y';\gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y'[U_2(0,y;\tau)+\chi_i(0,y;\tau)]\,\mathrm{d}\tau = O\{(\nu a)^{-2}(\nu l)^{-\frac{1}{2}}/[1+\surd(\nu(a-y))]\} \quad (0<y<a),$$

showing that this improved approximation satisfies the barrier condition even more accurately than $\hat{\chi}_0$.

A sufficiently accurate approximation procedure having been devised for calculating successive iterates of the scattered field, it is possible to seek an accurate estimate of the total scattered field. Since the source density corresponding to $U_1(x, y; \tau)$ is given by (8.38), the source density that generates $U_2(x, y; \tau)$ is readily derived from (8.45). The improved approximation to the iterate of the scattered field due to the field (8.34) is

$$\frac{\mathrm{i}}{\pi}\int_C P(\tau)\int_0^a \Delta(Y'; \gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y' U_2(x, y; \tau)\,\mathrm{d}\tau$$

and is generated by a source density $\Delta_1(y; \gamma)$ which is a better approximation to the iterate $\mu_1(y; \gamma)$ than $\hat{\Delta}_1(y; \gamma)$, given by (8.39). Thus

$$\mu_1(y; \gamma) \sim \Delta_1(y; \gamma) = \frac{1}{\pi}\int_C P(\tau)\,\mathrm{e}^{\mathrm{i}\nu a\sin\tau}\int_0^a \Delta(Y'; \gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y'$$
$$\times\left\{\left[\frac{\partial U(x, y-a; \tau)}{\partial x} - \frac{\partial U(x, -y-a; \tau)}{\partial x}\right]_{x=0+} + \frac{\mathrm{i}}{2\pi}\int_C\left[\frac{\partial U(x, y-a; \tau')}{\partial x} - \frac{\partial U(x, -y-a; \tau')}{\partial x}\right]_{x=0+}\right.$$
$$\left.\times\int_a^\infty\left[\frac{\partial U(x, -Y-a; \tau)}{\partial x}\right]_{x=0+}\mathrm{e}^{-\mathrm{i}\nu(Y-a)\sin\tau'}\,\mathrm{d}Y\,\mathrm{d}\tau'\right\}\mathrm{d}\tau \quad (0 < y < a). \tag{8.46}$$

The Y-integral is given by (8.42), and (8.40) implies that, for $0 < y < a$,

$$\left[\frac{\partial U(x, y-a; \tau)}{\partial x} - \frac{\partial U(x, -y-a; \tau)}{\partial x}\right]_{x=0+} = \left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\cos\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau\right)$$
$$\times\left\langle\frac{\mathrm{e}^{-\mathrm{i}\nu(a+y)}}{(a+y)^{\frac{1}{2}}}\{F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)] + F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)]\}\right.$$
$$\left.-\frac{\mathrm{e}^{-\mathrm{i}\nu(a-y)}}{(a-y)^{\frac{1}{2}}}\{F'[(2\nu)^{\frac{1}{2}}(a-y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)] + F'[(2\nu)^{\frac{1}{2}}(a-y)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)]\}\right\rangle. \tag{8.47}$$

It is then evident from the expressions (8.42) and (8.47) that a typical τ-integral in (8.46) is of the form

$$\frac{1}{\pi}\int_C P(\tau)\,\mathrm{e}^{\mathrm{i}\nu a\sin\tau}\sin(\nu Y'\sin\tau)\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)\{F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)]$$
$$+F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)]\}\,\mathrm{d}\tau$$
$$\sim\frac{2}{\pi}\int_S\sum_{n=1}^\infty[p_+(\tau)]^n\,\mathrm{e}^{-\mathrm{i}(2nl-a)\nu\sin\tau}\sin(\nu Y'\sin\tau)\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\tau)$$
$$\times F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau - \tfrac{1}{4}\pi)]\,\mathrm{d}\tau$$

by the methods of §3. This integral can now be evaluated by using (8.43) with relative error of order $(\nu l)^{-1}$ due solely to replacing $p_+(\tau)$ by its value at the saddle point, the result being

$$\frac{\mathrm{i}}{(\pi\nu l)^{\frac{1}{2}}}\sum_{n=1}^\infty[p(\tfrac{1}{2}\pi)]^n\left[\frac{\exp\{-\mathrm{i}\nu(2nl-a-Y') + \frac{1}{4}\pi\mathrm{i}\}}{\left(n - \frac{a+Y'}{2l}\right)^{\frac{1}{2}}\left[1 + \frac{a+y}{2nl-a-Y'}\right]} - \frac{\exp\{-\mathrm{i}\nu(2nl-a+Y') + \frac{1}{4}\pi\mathrm{i}\}}{\left(n - \frac{a-Y'}{2l}\right)^{\frac{1}{2}}\left[1 + \frac{a+y}{2nl-a+Y'}\right]}\right]$$

which to the leading two orders in a/l can be written as

$$-\frac{2\,\mathrm{e}^{\mathrm{i}(\nu a+\frac{1}{4}\pi)}}{(\pi\nu l)^{\frac{1}{2}}}\left\{\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l}, \tfrac{1}{2}, 1 - a/2l]\sin(\nu Y')\right.$$
$$\left.-\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l}, \tfrac{3}{2}, 1]\left[\frac{a+y}{2l}\sin(\nu Y') + \frac{\mathrm{i}Y'}{4l}\cos(\nu Y')\right]\right\},$$

where Φ is the generalized ζ-function defined by (3.23). These expansions in powers of the parameter a/l are 'forced' by the complicated denominators in the previous expression and, as in the semicircular case considered in §6, have the effect of the kernel of (8.46) being expressed in separable form.

On substituting (8.42) and (8.47) into (8.46) and estimating the τ-integrals as described above, it follows that

$$\Delta_1(y;\gamma) \sim e^{i\nu y}\left[c_1\left(\frac{2a}{a-y}\right)^{\frac{1}{2}}+d_1\left(\frac{a-y}{2a}\right)^{\frac{1}{2}}\right]-e^{-i\nu y}\left[c_1\left(\frac{2a}{a+y}\right)^{\frac{1}{2}}+d_1\left(\frac{a+y}{2a}\right)^{\frac{1}{2}}\right] -\frac{e^{-i(\nu a-\frac{1}{4}\pi)}}{2\pi^2}(c_1+d_1)\int_S L(y;\tau')\,d\tau', \quad (8.48)$$

where

$$L(y;\tau') = \left(\frac{2\pi}{\nu}\right)^{\frac{1}{2}}\left\{\frac{e^{-i\nu(a-y)}}{(a-y)^{\frac{1}{2}}}F'[(2\nu)^{\frac{1}{2}}(a-y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)]-\frac{e^{-i\nu(a+y)}}{(a+y)^{\frac{1}{2}}}F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi]\right\} \times\cos(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)/(1+\sin\tau'), \quad (8.49)$$

$$c_1 = \frac{1}{\pi(al)^{\frac{1}{2}}}\left\{i\Phi\left[p(\tfrac{1}{2}\pi)e^{-2i\nu l},\tfrac{1}{2},1-\frac{a}{2l}\right]\int_0^a \Delta(Y';\gamma)\sin(\nu Y')\,dY' +\frac{1}{4l}\Phi[p(\tfrac{1}{2}\pi)e^{-2i\nu l},\tfrac{3}{2},1]\int_0^a \Delta(Y';\gamma)\,Y'\cos(\nu Y')\,dY'\right\}, \quad (8.50a)$$

$$d_1 = -\frac{i}{\pi l}\left(\frac{a}{l}\right)^{\frac{1}{2}}\Phi[p(\tfrac{1}{2}\pi)e^{-2i\nu l},\tfrac{3}{2},1]\int_0^a \Delta(Y';\gamma)\sin(\nu Y')\,dY'. \quad (8.50b)$$

Since terms of order $(\nu a)^{-1}$ have been neglected, the τ'-integral in (8.48) is only significant near the edge $y = a$. One of the Y'-integrals in (8.50a, b) is given by (8.36), and the other can be evaluated similarly. As these expressions have been obtained without inserting the actual form of $\Delta(Y';\gamma)$, it is evident that the corresponding approximation $\Delta_2(y;\gamma)$ to $\mu_2(y;\gamma)$ is of the same form as (8.48) with the coefficients c_2, d_2 determined by replacing Δ by Δ_1 in (8.50a, b).

By elementary calculation, it readily follows from (8.48) that

$$\int_0^a \Delta_1(y;\gamma)\sin(\nu y)\,dy \sim 2ai(c_1+\tfrac{1}{3}d_1)-\tfrac{1}{2}i\,e^{i(2\nu a-\frac{1}{4}\pi)}\left(\frac{\pi a}{\nu}\right)^{\frac{1}{2}}c_1 -\frac{e^{-i(\nu a-\frac{1}{4}\pi)}}{2\pi^2}(c_1+d_1)\int_S\int_0^a L(y;\tau')\sin(\nu y)\,dy\,d\tau',$$

$$\int_0^a \Delta_1(y;\gamma)\,y\cos(\nu y)\,dy \sim \tfrac{2}{3}a^2(c_1-\tfrac{1}{5}d_1)+\tfrac{1}{2}a\,e^{i(2\nu a-\frac{1}{4}\pi)}\left(\frac{\pi a}{\nu}\right)^{\frac{1}{2}}c_1 -\frac{e^{i(\nu a-\frac{1}{4}\pi)}}{2\pi^2}(c_1+d_1)\int_S\int_0^a L(y;\tau')\,y\cos(\nu y)\,dy\,d\tau'.$$

Meanwhile, from (8.49)

$$\int_S\int_0^a L(y;\tau')\sin(\nu y)\,dy\,d\tau' = i\left(\frac{\pi}{2\nu}\right)^{\frac{1}{2}}\int_S \frac{\cos(\frac{1}{2}\tau'-\frac{1}{4}\pi)}{1+\sin\tau'}\int_{-a}^a \frac{e^{-i\nu y}}{(a+y)^{\frac{1}{2}}}(1-e^{-2i\nu y}) \times F'[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)]\,dy\,d\tau'$$

$$= i\frac{\pi^{\frac{1}{2}}}{\nu}e^{-i\nu a}\int_S \frac{\cos(\frac{1}{2}\tau'-\frac{1}{4}\pi)}{1+\sin\tau'}\left[\frac{F[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\frac{1}{2}\tau'-\frac{1}{4}\pi)]}{\sin(\frac{1}{2}\tau'-\frac{1}{4}\pi)}+\frac{e^{-2i\nu y}}{\sin\tau'} \times\{\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)F[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)]-F[(2\nu)^{\frac{1}{2}}(a+y)^{\frac{1}{2}}]\}\right]_{-a}^{a}d\tau'$$

$$= i\frac{\pi^{\frac{1}{2}}}{\nu}e^{-i\nu a}\int_S \frac{\cos(\frac{1}{2}\tau'-\frac{1}{4}\pi)}{1+\sin\tau'}\left\langle\frac{F[2(\nu a)^{\frac{1}{2}}\sin(\frac{1}{2}\tau'-\frac{1}{4}\pi)]-F(0)}{\sin(\frac{1}{2}\tau'-\frac{1}{4}\pi)}-\frac{e^{2i\nu a}}{\sin\tau'}[\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)-1]F(0) +\frac{e^{-2i\nu a}}{\sin\tau'}\{\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)F[2(\nu a)^{\frac{1}{2}}\sin(\tfrac{1}{2}\tau'-\tfrac{1}{4}\pi)]-F[2(\nu a)^{\frac{1}{2}}]\}\right\rangle d\tau',$$

the indefinite integral (8.20) having been used. Hence this double integral is of order ν^{-1} and therefore negligible in the formula

$$\int_0^a \Delta_1(y;\gamma)\sin(\nu y)\,\mathrm{d}y \sim 2a\mathrm{i}(c_1+\tfrac{1}{3}d_1)-\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(2\nu a-\frac{1}{4}\pi)}(\pi a/\nu)^{\frac{1}{2}}c_1. \tag{8.51a}$$

Similarly the other double integral is of order ν^{-1} and thus

$$\int_0^a \Delta_1(y;\gamma)\,y\cos(\nu y)\,\mathrm{d}y \sim \tfrac{2}{3}a^2(c_1-\tfrac{1}{5}d_1)+\tfrac{1}{2}a\,\mathrm{e}^{\mathrm{i}(2\nu a-\frac{1}{4}\pi)}(\pi a/\nu)^{\frac{1}{2}}c_1. \tag{8.51b}$$

Comparison of (8.39), (8.46) and (8.48) shows that only the cruder approximation $\hat{\Delta}_1(y;\gamma)$ contributes significantly to (8.51 a, b). Thus although each iterate contains additional terms as in (8.46), (8.48), such modifications do not affect, to the required order, subsequent iterates.

The simplest way to evaluate further iterates is to consider their sum. Setting

$$\mu^*(y;\gamma) = \sum_{n=1}^{\infty}\mu_n(y;\gamma) \sim \sum_{n=1}^{\infty}\Delta_n(y;\gamma) = \Delta^*(y;\gamma),$$

where $\mu^* = \mu-\mu_0$ as in (5.13), the approximation Δ^*, to the total source density μ^* required in addition to μ_0, must satisfy a Fredholm integral equation of the second kind, derived from the sequential relation of the form (8.46). Thus

$$\begin{aligned}\Delta^*(y;\gamma)-\Delta_1(y;\gamma) &= \frac{1}{\pi}\int_C P(\tau)\,\mathrm{e}^{\mathrm{i}\nu a\sin\tau}\int_0^a \Delta^*(Y';\gamma)\sin(\nu Y'\sin\tau)\,\mathrm{d}Y' \\ &\times\left\{\left[\frac{\partial U(x,y-a;\tau)}{\partial x}-\frac{\partial U(x,-y-a;\tau)}{\partial x}\right]_{x=0+}+\frac{\mathrm{i}}{2\pi}\int_C\left[\frac{\partial U(x,y-a;\tau')}{\partial x}-\frac{\partial U(x,-y-a;\tau')}{\partial x}\right]_{x=0+}\right. \\ &\left.\times\int_a^\infty\left[\frac{\partial U(x,-Y-a;\tau)}{\partial x}\right]_{x=0+}\mathrm{e}^{-\mathrm{i}\nu(Y-a)\sin\tau'}\,\mathrm{d}Y\,\mathrm{d}\tau'\right\}\mathrm{d}\tau \quad (0<y<a),\end{aligned}$$

and since the kernel involves, to the order required, the same functions of y as the 'forcing' function $\Delta_1(y;\gamma)$, this integral equation can be solved by direct substitution and equating of coefficients of such functions. Hence, corresponding to (8.48),

$$\begin{aligned}\Delta^*(y;\gamma) \sim\ & \mathrm{e}^{\mathrm{i}\nu y}\left[C^*\left(\frac{2a}{a-y}\right)^{\frac{1}{2}}+D^*\left(\frac{a-y}{2a}\right)^{\frac{1}{2}}\right]-\mathrm{e}^{-\mathrm{i}\nu y}\left[C^*\left(\frac{2a}{a+y}\right)^{\frac{1}{2}}+D^*\left(\frac{a+y}{2a}\right)^{\frac{1}{2}}\right] \\ &-\frac{\mathrm{e}^{-\mathrm{i}(\nu a-\frac{1}{4}\pi)}}{2\pi^2}(C^*+D^*)\int_S L(y;\tau')\,\mathrm{d}\tau',\end{aligned} \tag{8.52}$$

where, like (8.50 a, b),

$$\begin{aligned}C^*-c_1 &= \frac{1}{\pi(al)^{\frac{1}{2}}}\left\{\mathrm{i}\Phi\left[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1-\frac{a}{2l}\right]\int_0^a\Delta^*(Y';\gamma)\sin(\nu Y')\,\mathrm{d}Y'\right. \\ &\left.\quad+\frac{1}{4l}\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{3}{2},1]\int_0^a\Delta^*(Y';\gamma)\,Y'\cos(\nu Y')\,\mathrm{d}Y'\right\},\end{aligned}$$

$$D^*-d_1 = -\frac{\mathrm{i}}{\pi l}\left(\frac{a}{l}\right)^{\frac{1}{2}}\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{3}{2},1]\int_0^a\Delta^*(Y';\gamma)\sin(\nu Y')\,\mathrm{d}Y'.$$

The equations corresponding to (8.51 a, b) are

$$\int_0^a\Delta^*(y;\gamma)\sin(\nu y)\,\mathrm{d}y \sim 2a\mathrm{i}(C^*+\tfrac{1}{3}D^*)-\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(2\nu a-\frac{1}{4}\pi)}\left(\frac{\pi a}{\nu}\right)^{\frac{1}{2}}C^*, \tag{8.53}$$

$$\int_0^a\Delta^*(y;\gamma)\,y\cos(\nu y)\,\mathrm{d}y \sim \tfrac{2}{3}a^2(C^*-\tfrac{1}{5}D^*)+\tfrac{1}{2}a\,\mathrm{e}^{\mathrm{i}(2\nu a-\frac{1}{4}\pi)}\left(\frac{\pi a}{\nu}\right)^{\frac{1}{2}}C^*,$$

and it is only necessary to neglect terms of order $(\nu a)^{-1}$ and a^3/l^3 to write the solution of the simultaneous equations for C^* and D^* as

$$(C^*, D^*) \sim \left\{1+\left[\frac{2}{\pi}\left(\frac{a}{l}\right)^{\frac{1}{2}}-\frac{e^{i(2\nu a-\frac{1}{4}\pi)}}{2(\pi\nu l)^{\frac{1}{2}}}\right]\Phi\left[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{1}{2}, 1-\frac{a}{2l}\right]\right.$$
$$\left.-\left[\frac{5}{6\pi}+\frac{e^{i(2\nu a-\frac{1}{4}\pi)}}{8(\pi\nu a)^{\frac{1}{2}}}\right]\left(\frac{a}{l}\right)^{\frac{3}{2}}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{3}{2}, 1]\right\}^{-1}$$
$$\times\frac{1}{\pi(al)^{\frac{1}{2}}}\left\{i\Phi\left[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{1}{2}, 1-\frac{a}{2l}\right]\int_0^a \Delta(y;\gamma)\sin(\nu y)\,dy\right.$$
$$+\frac{1}{4l}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{3}{2}, 1]\int_0^a \Delta(y;\gamma)\,y\cos(\nu y)\,dy,$$
$$\left.-\frac{ia}{l}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{3}{2}, 1]\int_0^a \Delta(y;\gamma)\sin(\nu y)\,dy\right\}. \qquad (8.54)$$

Now by differentiation of (8.24) with respect to τ and comparison with (8.36), it may be shown that

$$\int_0^a \Delta(y;\gamma)\,y\cos(\nu y)\,dy = ia\int_0^a \Delta(y;\gamma)\sin(\nu y)\,dy$$
$$+2\left(\frac{a}{\nu\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}\pi i}\left[\frac{e^{-i\nu a(1+\sin\gamma)}}{1+\sin\gamma}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)-\frac{e^{-i\nu a(1-\sin\gamma)}}{1-\sin\gamma}\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)\right]+O\left(\frac{1}{\nu}\right). \qquad (8.55)$$

Also the definition (3.23) allows the corrections of order a/l to be absorbed in the Φ-functions, for example

$$\Phi(z, \tfrac{1}{2}, 1-a/2l)+(a/4l)\,\Phi(z, \tfrac{3}{2}, 1) \sim \Phi(z, \tfrac{1}{2}, 1-a/l).$$

Hence, when terms of order $a(\nu l^3)^{-\frac{1}{2}}$ are also neglected, (8.54) can be simplified to the form

$$(C^*, D^*) \sim \left\{1+\frac{2}{\pi}\left(\frac{a}{l}\right)^{\frac{1}{2}}\Phi\left[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{1}{2}, 1+\frac{a}{3l}\right]-\frac{e^{i(2\nu a-\frac{1}{4}\pi)}}{2(\pi\nu l)^{\frac{1}{2}}}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{1}{2}, 1]\right\}^{-1}$$
$$\times\frac{i}{\pi(al)^{\frac{1}{2}}}\int_0^a \Delta(y;\gamma)\sin(\nu y)\,dy\left\{\Phi\left[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{1}{2}, 1-\frac{a}{l}\right], -\frac{a}{l}\Phi[p(\tfrac{1}{2}\pi)\,e^{-2i\nu l}, \tfrac{3}{2}, 1],\right\} \qquad (8.56)$$

where the integral is given by (8.36).

(d) *The total scattered field at the line of the barrier*

These results are more accurate than those of §6 because, after sufficiently accurate approximations to the scattered fields have been constructed, the focusing of the rays on the offshore line $x = 0$, $a < y < l$ is more pronounced for the barrier than for the semicircular obstacle. Consequently the form of the source density iterates is established at μ_1 (all orders in a/l) for the barrier, but only at μ_2 (leading order in a/l) for the semicircle (equation (6.19)). The appearance of two constants above corresponds to the retention of the two leading orders in a/l in (8.48).

Another consequence of the barrier geometry is a simple physical interpretation of the source density. Substitution of (2.20) in (5.2) shows that

$$\mu(y;\gamma) = e^{-by}[\partial\psi/\partial x]_{x=0-}^{x=0+}$$

since ψ is continuous across the barrier. Subsequent use of (2.3) and (2.5) shows that the discontinuity across the barrier of the offshore velocity component v is given by

$$[v]_{x=0-}^{x=0+} = -[h(0)]^{-1}\mu(y;\gamma)\,\mathrm{e}^{-(by+\mathrm{i}\omega t)} \quad (0 < y < a),$$

which is consistent with the physical interpretation of μ as a vorticity distribution. The above constructions of scattered fields are such that the approximation of the components μ_0, μ^* of μ by the respective source densities Δ, Δ^* has far greater accuracy than can be reasonably achieved in the subsequent calculation of Δ and Δ^*.

The scattered field due to the source density $\Delta(y;\gamma)$ has been discussed above in detail and illustrated by figure 4. The field $\hat{\chi}_0$, corresponding to the limit $bl \to \infty$, is given by (8.15) and is asymptotically the same as χ^*, illustrated in figure 3. In particular, it follows from (8.1), (8.8) and (8.9) that on the line of the barrier

$$\hat{\chi}_0(0,y;\gamma) \sim \chi^*(0,y;\gamma) = -(1/\pi^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}[\nu(y-a)+\frac{1}{4}\pi]}$$
$$\times\{F[(2\nu)^{\frac{1}{2}}(y-a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)]\,\mathrm{e}^{\mathrm{i}\nu a\sin\gamma} - F[(2\nu)^{\frac{1}{2}}(y-a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu a\sin\gamma}\} \quad (y > a)$$

which, with the aid of (8.4), is seen to be a sum of edge-diffracted waves. The additional field, due to bl being finite, can be estimated from (3.24), but the larger values of y mean that less simplification is possible than in (8.35). Thus, to leading order only

$$\int_0^a \Delta(Y;y)\,G_\chi^*(0,y;Y)\,\mathrm{d}Y \sim \frac{\mathrm{i}}{\pi}\int_S P_+(\tau)\sin(\nu y\sin\tau)\int_0^a \Delta(Y;\gamma)\sin(\nu Y\sin\tau)\,\mathrm{d}Y\,\mathrm{d}\tau$$
$$\sim \frac{p(\frac{1}{2}\pi)\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}}{2(\pi\nu l)^{\frac{1}{2}}}\,\mathrm{e}^{-2\mathrm{i}\nu l}\int_0^a \Delta(Y;\gamma)\sin(\nu Y)\,\mathrm{d}Y$$
$$\times\{\mathrm{e}^{\mathrm{i}\nu y}\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1-y/2l] - \mathrm{e}^{-\mathrm{i}\nu y}\Phi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1+y/2l]\} \quad (0 < y < l) \qquad (8.57)$$

which indicates waves of slowly varying amplitude travelling to and from the shore along the line of the barrier.

The exponential factors $\mathrm{e}^{\pm\mathrm{i}\nu y}$ in (8.52) show that, as expected from the construction, only on the line of the barrier can the scattered field due to the source density $\Delta^*(y;\gamma)$ be comparable with that due to $\Delta(y;\gamma)$. Since (8.49) and (8.52) imply that Δ^* is an odd function of y, it is possible to write

$$\int_0^a \Delta^*(Y;\gamma)\,G_\chi^\circ(0,y;Y)\,\mathrm{d}Y = \tfrac{1}{4}\mathrm{i}\int_{-a}^a \Delta^*(Y;\gamma)\,H_0^{(2)}[\nu(y-Y)]\,\mathrm{d}Y \quad (y > a). \qquad (8.58)$$

The contribution of the term $C^*\mathrm{e}^{\mathrm{i}\nu y}[2a/(a-y)]^{\frac{1}{2}}$ in Δ^* to (8.58) can be estimated by comparison with the derivative on the barrier of the field (8.6). The result is $\mathrm{e}^{-\mathrm{i}\nu(y-a)}F[(2\nu)^{\frac{1}{2}}(y-a)^{\frac{1}{2}}]$ times a multiple of order $(\nu l)^{-\frac{1}{2}}$ plus a much smaller term due to replacing the limit of integration by $-\infty$. The contribution of the term $D^*\mathrm{e}^{\mathrm{i}\nu y}[(a-y)/2a]^{\frac{1}{2}}$ in Δ^* to (8.58) is found by considering also the field obtained by differentiating (8.1) twice with respect to τ and setting $\tau = \frac{1}{2}\pi$. A similar result emerges. The first two terms of (8.52) produce smaller edge-diffracted waves in (8.58) because they are due to waves travelling shoreward. The same is not true of the next two terms in (8.52) which were inserted by reflexion in the shoreline without taking into account the edge at $(0, a)$ and which now contribute terms in (8.58) that are of the same order as (8.57). This is because, on inserting with negligible error the asymptotic form of $H_0^{(2)}$, the exponential dependence of the integrand on Y cancels, leaving

$$-\tfrac{1}{4}\mathrm{i}\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\nu Y}\left[C^*\left(\frac{2a}{a+Y}\right)^{\frac{1}{2}}+D^*\left(\frac{a+Y}{2a}\right)^{\frac{1}{2}}\right]H_0^{(2)}[\nu(y-Y)]\,\mathrm{d}Y$$
$$\sim\frac{\mathrm{e}^{-\mathrm{i}(\nu y+\frac{1}{4}\pi)}}{2(2\pi\nu)^{\frac{1}{2}}}\int_{-a}^{a}\left[C^*\left(\frac{2a}{a+Y}\right)^{\frac{1}{2}}+D^*\left(\frac{a+Y}{2a}\right)^{\frac{1}{2}}\right]\frac{\mathrm{d}Y}{(y-Y)^{\frac{1}{2}}}\quad(y>a)$$
$$=\frac{1}{2}\left(\frac{a}{\pi\nu}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}(\nu y+\frac{1}{4}\pi)}\left\{\left[2C^*+D^*\left(\frac{y+a}{2a}\right)\right]\arcsin\left(\frac{2a}{y+a}\right)^{\frac{1}{2}}-D^*\left(\frac{y-a}{2a}\right)^{\frac{1}{2}}\right\},$$

where C^* and D^* are given by (8.56). With L defined by (8.49), the contribution of the integral in (8.52)–(8.58) is evidently negligible. Also generated by $\varDelta^*$ on the line of the barrier is a field like (8.57) but with $\varDelta$ replaced by $\varDelta^*$ throughout. The $\varDelta^*$-integral is related to that involving $\varDelta$ by (8.53) and (8.56), whence

$$\int_0^a[\varDelta(Y;\gamma)+\varDelta^*(Y;\gamma)]\,G_\chi^*(0,y;Y)\,\mathrm{d}Y\sim\frac{\int_0^a\varDelta(Y;\gamma)\,G_\chi^*(0,y;Y)\,\mathrm{d}Y}{1+2\pi^{-1}(a/l)^{\frac{1}{2}}\varPhi[p(\frac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\frac{1}{2},1]}\quad(0<y<l).$$

On collecting terms, the total scattered field at $x=0$, $a<y<l$ is found to be given by

$$\chi(0,y;\gamma)=\int_0^a\mu(Y;\gamma)\,G_\chi(0,y;Y)\,\mathrm{d}Y\sim\int_0^a[\varDelta(Y;\gamma)+\varDelta^*(Y;\gamma)]\,G_\chi(0,y;Y)\,\mathrm{d}Y$$
$$\sim-(1/\pi^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}[\nu(y-a)+\frac{1}{4}\pi]}\{F[(2\nu)^{\frac{1}{2}}(y-a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\gamma)]\,\mathrm{e}^{\mathrm{i}\nu a\sin\gamma}$$
$$-F[(2\nu)^{\frac{1}{2}}(y-a)^{\frac{1}{2}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)]\,\mathrm{e}^{-\mathrm{i}\nu a\sin\gamma}\}$$
$$+\frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}}{(\pi\nu l)^{\frac{1}{2}}}\frac{\int_0^a\varDelta(Y;\gamma)\sin(\nu Y)\,\mathrm{d}Y}{1+2\pi^{-1}(a/l)^{\frac{1}{2}}\varPhi[p(\frac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\frac{1}{2},1]}\left[\frac{1}{\pi}\mathrm{e}^{-\mathrm{i}\nu y}\arcsin\left(\frac{2a}{y+a}\right)^{\frac{1}{2}}\varPhi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1]\right.$$
$$\left.+\tfrac{1}{2}p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l}\{\mathrm{e}^{\mathrm{i}\nu y}\varPhi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1-y/2l]-\mathrm{e}^{-\mathrm{i}\nu y}\varPhi[p(\tfrac{1}{2}\pi)\,\mathrm{e}^{-2\mathrm{i}\nu l},\tfrac{1}{2},1+y/2l]\}\right]$$
$$(a<y<l).\quad(8.59)$$

Further, it is evident that

$$[\partial\chi(x,y;\gamma)/\partial x]_{x=0}\sim 0\quad(a<y<l),\quad(8.60)$$

since the only way that a non-zero contribution can arise is through $p(\tau)$ not being symmetric about $\tau=\frac{1}{2}\pi$. Then (2.20) implies

$$\frac{\partial\psi}{\partial x}(0,y;\gamma)\sim\frac{\mathrm{i}b}{\sigma}\mathrm{e}^{by}\chi(0,y;\gamma),\quad\frac{\partial\psi(0,y;\gamma)}{\partial y}\sim\frac{\partial[\mathrm{e}^{by}\chi(0,y;\gamma)]}{\partial y}\quad(a<y<l).$$

On substitution of (8.59), it follows since $|b\pm\mathrm{i}\nu|=(b^2+\nu^2)^{\frac{1}{2}}=b/\sigma$ that corresponding terms in these ψ-derivatives only differ in phase. In the context of (2.3), it is seen that the alongshore and offshore components, at the line of the barrier, of the velocity field due to scattering are such that, to leading order, corresponding terms differ only by constant phases.

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